Cop and robber games when the robber can hide and ride

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Abstract. In the classical cop and robber game, two players, the cop $\mathcal{C}$ and the robber $\mathcal{R}$, move alternatively along edges of a finite graph $G = (V, E)$. The cop captures the robber if both players are on the same vertex at the same moment of time. A graph G is called *cop win* if the cop always captures the robber after a finite number of steps. Nowakowski, Winkler (1983) and Quilliot (1983) characterized the cop-win graphs as graphs admitting a dismantling scheme. In this paper, we characterize in a similar way the class $\mathcal{CWFR}(s, s')$ of cop-win graphs in the game in which the cop and the robber move at different speeds s' and s, $s' \leq s$. We also establish some connections between cop-win graphs for this game with $s' < s$ and Gromov's hyperbolicity. In the particular case $s' = 1$ and $s = 2$, we prove that the class of cop-win graphs is exactly the well-known class of dually chordal graphs. We show that all classes $\mathcal{CWFR}(s, 1)$, $s \geq 3$, coincide and we provide a structural characterization of these graphs. We also investigate several dismantling schemes necessary or sufficient for the cop-win graphs in the game in which the robber is visible only every k moves for a fixed integer $k > 1$. We characterize the graphs which are cop-win for any value of k. Finally, we consider the game where the cop wins if he is at distance at most 1 from the robber and we characterize via a specific dismantling scheme the bipartite graphs where a single cop wins in this game.

Keywords: Cop and robber games, cop-win graphs, dismantling orderings, δ-hyperbolicity.

1. Introduction

1.1. **The cop and robber game(s).** The cop and robber game originated in the 1980's with the work of Nowakowski, Winkler [30], Quilliot [31], and Aigner, Fromme [2], and since then has been intensively investigated by numerous authors and under different names (e.g., hunter and rabbit game [27]). Cop and robber is a pursuit-evasion game played on finite undirected graphs. Player cop $\mathcal{C}$ has one or several cops who attempt to capture the robber $\mathcal{R}$. At the beginning of the game, $\mathcal{C}$ occupies vertices for the initial position of his cops, then $\mathcal{R}$ occupies another vertex. Thereafter, the two sides move alternatively, starting with $\mathcal{C}$, where a move is to slide along an edge or to stay at the same vertex, i.e. pass. Both players have full knowledge of the current positions of their adversaries. The objective of $\mathcal{C}$ is to capture $\mathcal{R}$, i.e., to be at some moment of time, or *step*, at the same vertex as the robber. The objective of $\mathcal{R}$ is to continue evading the cop. A *cop-win graph* [2, 30, 31] is a graph in

which a *single cop* captures the robber after a finite number of moves for all possible initial positions of $\mathcal{C}$ and $\mathcal{R}$. Denote by $\mathcal{CW}$ the set of all cop-win graphs. The cop-number of a graph G, introduced by Aigner and Fromme [2], is the minimum number of cops necessary to capture the robber in G. Different combinatorial (lower and upper) bounds on the cop number for different classes of graphs were given in [2, 4, 9, 17, 22, 32, 33, 34] (see also the survey paper [3] and the annotated bibliography [21]).

In this paper, we investigate the cop-win graphs for three basic variants of the classical cop and robber game (for continuous analogous of these games, see [21]). In the *cop and fast robber game*, introduced by Fomin, Golovach, and Kratochvil [19] and further investigated in [29] (see also [20]), the cop is moving at unit speed while the speed of the robber is an integer $s \geq 1$ or is unbounded ($s \in \mathbb{N} \cup \{\infty\}$), i.e., at his turn, $\mathcal{R}$ moves along a path of length at most s which does not contain vertices occupied by $\mathcal{C}$. Let $\mathcal{CWFR}(s)$ denote the class of all graphs in which a single cop having speed 1 captures a robber having speed s. Obviously, $\mathcal{CWFR}(1) = \mathcal{CW}$. In a more general version, we will suppose that $\mathcal{R}$ moves with speed s and $\mathcal{C}$ moves with speed $s' \leq s$ (if $s' > s$, then the cop can always capture the robber by strictly decreasing at each move his distance to the robber). We will denote the class of cop-win graphs for this version of the game by $\mathcal{CWFR}(s, s')$. A *witness version* of the cop and robber game was recently introduced by Clarke [18]. In this game, the robber has unit speed and moves by having perfect information about cop positions. On the other hand, the cop no longer has full information about robber's position but receives it only occasionally, say every k units of time, in which case, we say that $\mathcal{R}$ is *visible* to $\mathcal{C}$, otherwise, $\mathcal{R}$ is *invisible* (this kind of constraint occurs, for instance, in the "Scotland Yard" game [14]). Following [18], we call a graph G *k-winnable* if a single cop can guarantee a win with such witness information and denote by $\mathcal{CWW}(k)$ the class of all k-winnable graphs. Notice that $\mathcal{CWFR}(s) \subseteq \mathcal{CWW}(s)$ because the first game can be viewed as a particular version of the second game in which $\mathcal{C}$ moves only at the turns when he receives the information about $\mathcal{R}$. Finally, the game of *distance k cop and robber* introduced by Bonato and Chiniforooshan [11] is played in the same way as classical cop and robber, except that the cop wins if a cop is within distance at most k from the robber (following the name of an analogous game in continuous spaces [21], we will refer to this game as *cop and robber with radius of capture k*). We denote by $\mathcal{CWRC}(k)$ the set of all cop-win graphs in this game.

1.2. **Cop-win graphs.** Cop-win graphs (in $\mathcal{CW}$) have been characterized by Nowakowski and Winkler [30], and Quillot [32] (see also [2]) as dismantlable graphs (see Section 1.4 for formal definitions). Let $G = (V, E)$ be a graph and u, v two vertices of G such that any neighbor of v (including v itself) is also a neighbor of u. Then there is a retraction of G to $G \setminus \{v\}$ taking v to u. Following [25], we call this retraction a *fold* and we say that v is *dominated* by u. A graph G is *dismantlable* if it can be reduced, by a sequence of folds, to a single vertex. In other words, an n-vertex graph G is dismantlable if its vertices can be ordered $v_1, \ldots, v_n$ so that for each vertex $v_i, 1 \leq i < n$, there exists another vertex v_j with $j > i$, such that $N_1(v_i) \cap X_i \subseteq N_1(v_j)$, where $X_i := \{v_i, v_{i+1}, \ldots, v_n\}$ and $N_1(v)$ denotes

the closed neighborhood of v. For a simple proof that dismantlable graphs are the cop-win graphs, see the book [25]. An alternative (more algorithmic) proof of this result is given in [27]. Dismantlable graphs include bridged graphs (graphs in which all isometric cycles have length 3) and Helly graphs (absolute retracts) [6, 25] which occur in several other contexts in discrete mathematics. Except the cop and robber game, dismantlable graphs are used to model physical processes like phase transition [13], while bridged graphs occur as 1-skeletons of systolic complexes in the intrinsic geometry of simplicial complexes [15, 24, 26]. Dismantlable graphs are closed under retracts and direct products, i.e., they constitute a variety [30].

1.3. **Our results.** In this paper, we characterize the graphs of the class $\mathcal{CWFR}(s,s')$ for all speeds s, s' in the same vein as cop-win graphs, by using a specific dismantling order. Our characterization allows to decide in polynomial time if a graph G belongs to any of considered classes $\mathcal{CWFR}(s,s')$. In the particular case $s' = 1$, we show that $\mathcal{CWFR}(2)$ is exactly the well-known class of dually chordal graphs. Then we show that the classes $\mathcal{CWFR}(s)$ coincide for all $s \geq 3$ and that the graphs G of these classes have the following structure: the block-decomposition of G can be rooted in such a way that any block has a dominating vertex and that for each non-root block, this dominating vertex can be chosen to be the articulation point separating the block from the root. We also establish some connections between the graphs of $\mathcal{CWFR}(s,s')$ with $s' < s$ and Gromov's hyperbolicity. More precisely, we prove that any δ-hyperbolic graph belongs to the class $\mathcal{CWFR}(2r, r+2\delta)$ for any $r > 0$, and that, for any $s \geq 2s'$, the graphs in $\mathcal{CWFR}(s,s')$ are $(s-1)$-hyperbolic. We also establish that Helly graphs and bridged graphs belonging to $\mathcal{CWFR}(s,s')$ are s^2-hyperbolic and we conjecture that, in fact all graphs of $\mathcal{CWFR}(s,s')$, where $s' < s$, are δ-hyperbolic, where δ depends only of s.

In the second part of our paper, we characterize the graphs that are s-winnable for all s (i.e., graphs in $\cap_{s\geq 1}\mathcal{CWW}(s)$) using a similar decomposition as for the graphs from the classes $\mathcal{CWFR}(s)$, $s \geq 3$. On the other hand, we show that for each s, $\mathcal{CWW}(s) \setminus \mathcal{CWW}(s+1)$ is non-empty , contrary to the classes $\mathcal{CWFR}(s)$. We show that all graphs of $\mathcal{CWW}(2)$, i.e., the 2-winnable graphs, have a special dismantling order (called bidismantling), which however does not ensure that a graph belongs to $\mathcal{CWW}(2)$. We present a stronger version of bidismantling and show that it is sufficient for ensuring that a graph is 2-winnable. We extend bidismantling to any $k \geq 3$ and prove that for all odd k, bidismantling is sufficient to ensure that $G \in \mathcal{CWFR}(k)$. Finally, we characterize the bipartite members of $\mathcal{CWRC}(1)$ via an appropriate dismantling scheme. We also formulate several open questions.

1.4. **Preliminaries.** For a graph $G = (V,E)$ and a subset X of its vertices, we denote by $G(X)$ the subgraph of G induced by X. We will write $G \setminus \{x\}$ and $G \setminus \{x,y\}$ instead of $G(V \setminus \{x\})$ and $G(V \setminus \{x,y\})$. The *distance* $d(u,v) := d_G(u,v)$ between two vertices u and v of a graph G is the length (number of edges) of a shortest (u,v)-path. An induced subgraph H of G is *isometric* if the distance between any pair of vertices in H is the same as that in G. The *ball* (or disk) $N_r(x)$ of center x and radius $r \geq 0$ consists of all vertices of G at distance at most r from x. In particular, the unit ball $N_1(x)$ comprises x and the neighborhood $N(x)$. The

punctured ball $N_r(x, G \setminus \{y\})$ of center x, radius r, and puncture y is the set of all vertices of G which can be connected to x by a path of length at most r avoiding the vertex y, i.e., this is the ball of radius r centered at x in the graph $G \setminus \{y\}$. A *retraction* φ of a graph $H = (W, F)$ is an idempotent nonexpansive mapping of H into itself, that is, $\varphi^2 = \varphi : W \to W$ with $d(\varphi(x), \varphi(y)) \leq d(x, y)$ for all $x, y \in W$. The subgraph of H induced by the image of H under φ is referred to as a *retract* of H.

A *strategy* for the cop is a function σ which takes as an input the first i moves of both players and outputs the $(i+1)$th move c_{i+1} of the cop. A strategy for the robber is defined in a similar way. A cop's strategy σ is *winning* if for any sequence of moves of the robber, the cop, following σ, captures the robber after a finite sequence of moves. Note that if the cop has a winning strategy σ in a graph G, then there exists a winning strategy σ' for the cop that only depends of the last positions of the two players (such a strategy is called *positional*). This is because cop and robber games are parity games (by considering the directed graph of configurations) and parity games always admit positional strategies for the winning player [28]. A strategy for the cop is called *parsimonious* if at his turn, the cop captures the robber (in one move) whenever he can. For example, in the cop and fast robber game, at his move, the cop following a parsimonious strategy always captures a robber located at distance at most s' from his current position. It is easy to see that in the games investigated in this paper, if the cop has a (positional) winning strategy, then he also has a parsimonious (positional) winning strategy.

2. Cop-win graphs for game with fast robber: class $\mathcal{CWFR}(s, s')$

In this section, first we characterize the graphs of $\mathcal{CWFR}(s, s')$ via a specific dismantling scheme, allowing to recognize them in polynomial time. Then we show that any δ-hyperbolic graph belongs to the class $\mathcal{CWFR}(2r, r + 2\delta)$ for any $r \geq 1$. We conjecture that the converse is true, i.e., any graph from $\mathcal{CWFR}(s, s')$ with $s' < s$ is δ-hyperbolic for some value of δ depending only of s, and we confirm this conjecture in several particular cases.

2.1. **Graphs of $\mathcal{CWFR}(s, s')$.** For technical convenience, we will consider a slightly more general version of the game: given a subset of vertices X of a graph $G = (V, E)$, the *X-restricted game* with cop and robber having speeds s' and s, respectively, is a variant in which $\mathcal{C}$ and $\mathcal{R}$ can pass through any vertex of G but can stand only at vertices of X (i.e., the beginning and the end of each move are in X). A subset of vertices X of a graph $G = (V, E)$ is (s, s')-*winnable* if the cop captures the robber in the X-restricted game. In the following, given a subset X of admissible positions, we say that a sequence of vertices $S_r = (a_1, \ldots, a_p, \ldots)$ of a graph $G = (V, E)$ is *X-valid* for a robber with speed s (respectively, for a cop with speed s') if, for any k, we have $a_k \in X$ and $d(a_{k-1}, a_k) \leq s$ (respectively, $d(a_{k-1}, a_k) \leq s'$). We will say that a subset of vertices X of a graph $G = (V, E)$ is (s, s')-*dismantlable* if the vertices of X can be ordered $v_1, \ldots, v_m$ in such a way that for each vertex $v_i, 1 \leq i < m$, there exists another vertex v_j with $j > i$, such that $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_{s'}(v_j)$, where

$X_i := \{v_i, v_{i+1}, \ldots, v_m\}$ and $X_m = \{v_m\}$. A graph $G = (V, E)$ is (s, s')-*dismantlable* if its vertex-set V is (s, s')-dismantlable.

Theorem 1. *For any $s, s' \in \mathbb{N} \cup \{\infty\}$, $s' \leq s$, a graph $G = (V, E)$ belongs to the class $\mathcal{CWFR}(s, s')$ if and only if G is (s, s')-dismantlable.*

Proof. First, suppose that G is (s, s')-dismantlable and let $v_1, \ldots, v_n$ be an (s, s')-dismantling ordering of G. By induction on $n - i$ we will show that for each level-set $X_i = \{v_i, \ldots, v_n\}$ the cop captures the robber in the X_i-restricted game. This is obviously true for $X_n = \{v_n\}$. Suppose that our assertion is true for all sets $X_n, \ldots, X_{i+1}$ and we will show that it still holds for X_i. Let $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_{s'}(v_j)$ for a vertex $v_j \in X_i$. Consider a parsimonious positional winning strategy σ_{i+1} for the cop in the X_{i+1}-restricted game. We build a parsimonious winning strategy σ_i for the cop in the X_i-restricted game: the intuitive idea is that if the cop sees the robber in v_i, he plays as in the X_{i+1}-restricted game when the robber is in v_j. Let σ_i be the strategy for the X_i-restricted game defined as follows. For any positions $c \in X_i$ of the cop and $r \in X_i$ of the robber, set $\sigma_i(c, r) = r$ if $d(c, r) \leq s'$, otherwise $\sigma_i(c, r) = \sigma_{i+1}(c, r)$ if $c, r \neq v_i$, $\sigma_i(c, v_i) = \sigma_{i+1}(c, v_j)$ if $c \notin \{v_i, v_j\}$, and $\sigma_i(v_i, r) = v_j$ if $r \neq v_i$ (in fact, if the cop plays σ_i he will never move to v_i except to capture the robber there). By construction, the strategy σ_i is parsimonious; in particular, $\sigma_i(v_j, v_i) = v_i$, because $d(v_i, v_j) \leq s'$. We now prove that σ_i is winning.

Consider any X_i-valid sequence $S_r = (r_1, \ldots, r_p, \ldots)$ of moves of the robber and any trajectory $(\pi_1, \ldots, \pi_p, \ldots)$ extending S_r, where π_p is a simple path of length at most s from r_p to r_{p+1} along which the robber moves. Let $S'_r = (r'_1, \ldots r'_p, \ldots)$ be the sequence obtained by setting $r'_k = r_k$ if $r_k \neq v_i$ and $r'_k = v_j$ if $r_k = v_i$. For each p, set $\pi'_p = \pi_p$ if $v_i \notin \{r_p, r_{p+1}\}$. If $v_i = r_{p+1}$ (resp. $v_i = r_p$), set π'_p be a shortest path from r_p to v_j (resp. from v_j to r_{p+1}) if π_p does not contain v_j and set π'_p be the subpath of π_p between r_p and v_j (resp. between v_j and r_{p+1}) otherwise. Since $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_{s'}(v_j)$, we infer that S'_r is a X_{i+1}-valid sequence of moves for the robber. By induction hypothesis, for any initial location of $\mathcal{C}$ in X_{i+1}, the strategy σ_{i+1} allows the cop to capture the robber which moves according to S'_r in the X_{i+1}-restricted game. Let c'_{m+1} be the position of the cop after his last move and $S'_c = (c'_1, \ldots, c'_{m+1})$ be the sequence of positions of the cop in the X_{i+1}-restricted game against S'_r using σ_{i+1}. Let $S_c = (c_1, \ldots, c_p, \ldots)$ be the sequence of positions of the cop in the X_i-restricted game against S_r using σ_i. From the definition of S'_r and σ_i, S_c and S'_c coincide at least until step m, i.e., $c'_k = c_k$ for $k = 1, \ldots, m$. Moreover, if $c'_{m+1} \neq c_{m+1}$ then $c_{m+1} = r_m = v_i$ and $c'_{m+1} = r'_m = v_j$. In the X_{i+1}-restricted version of the game, the robber is captured, either (i) because after his last move, his position r'_m is at distance at most s' from cop's current position c'_m, or (ii) because his trajectory π'_m from r'_m to r'_{m+1} passes via c'_{m+1}.

In case (i), since $d(r'_m, c'_m) \leq s'$ and the strategy σ_{i+1} is parsimonious, we conclude that $c'_{m+1} = r'_m$. If $c'_{m+1} = r'_m \neq v_j$, then from the definition of S'_r and σ_i, we conclude that $c_{m+1} = c'_{m+1} = r'_m = r_m$, whence $c_{m+1} = r_m$ and $\mathcal{C}$ captures $\mathcal{R}$ using σ_i. Now suppose that $c'_{m+1} = r'_m = v_j$. If $r_m = v_j$, then $d(c_m, r_m) \leq s'$ because $c_m = c'_m$ and thus $\mathcal{C}$ captures $\mathcal{R}$ at

v_j using σ_i. On the other hand, if $r_m = v_i$, either $c_{m+1} = v_i$ and we are done, or $c_{m+1} = v_j$ and since $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_{s'}(v_j)$, the robber is captured at the next move of the cop, i.e., $c_{m+2} = r_{m+1}$ holds.

In case (ii), either the path π'_m from r'_m to r'_{m+1} is a subpath of π_m, or $v_i \in \{r_m, r_{m+1}\}$ and π_m does not go via v_j. In the first case, note that $c_{m+1} = c'_{m+1}$, otherwise $c_{m+1} = v_i = r_m$ by construction of σ_i and thus the robber has been captured before. Therefore the trajectory π_m of the robber in the X_i-game traverses the position c_{m+1} of the cop and we are done. Now suppose that π_m does not go via v_j and $v_i \in \{r_m, r_{m+1}\}$. Note that in this case, $c_{m+1} = c'_{m+1}$ holds; otherwise, $c'_{m+1} = r'_m = v_j$ and $c_{m+1} = r_m = v_i$ and therefore, the robber is caught at step $m+1$. If c_{m+1} belongs to π_m, then we are done as in the first case. So suppose that $c_{m+1} \notin \pi_m$. If $r_{m+1} = v_i$, then $r_m \in N_s(v_i, G \setminus \{v_j\}) \subseteq N_{s'}(v_j)$. Since, π'_m is a shortest path and c'_{m+1} belongs to this path, $d(c'_{m+1}, v_j) \leq s'$ and thus either $c_{m+2} = v_i = r_{m+1}$ if $d(c'_{m+1}, v_i) \leq s'$, or $c_{m+2} = v_j$ since σ_{i+1} is parsimonious. In the latter case, since $N_s(v_i, G \setminus \{v_j\}) \subseteq N_{s'}(v_j)$, $r_{m+1} = v_i$, and $c_{m+2} = v_j$, the robber will be captured at the next move. Finally, suppose that $r_m = v_i$. Then $r'_m = v_j$. Since π_m is a path of length at most s avoiding v_j, we conclude that $r_{m+1} \in N_s(v_i, G \setminus \{v_j\}) \subseteq N_{s'}(v_j)$. Since π'_m is a shortest path from v_j to r_{m+1} containing the vertex $c'_{m+1} = c_{m+1}$, we have $d(c_{m+1}, r_{m+1}) \leq d(v_j, r_{m+1}) \leq s'$. Therefore, the cop captures the robber in r_{m+1} at his next move, i.e., $c_{m+2} = r_{m+1}$. This shows that a (s, s')-dismantlable graph G belongs to $\mathcal{CWFR}(s, s')$.

Conversely, suppose that for a X-restricted game played on a graph $G = (V, E)$ there is a positional winning strategy σ for the cop. We assert that X is (s, s')-dismantlable. This is obviously true if X contains a vertex y such that $d(y, x) \leq s'$ for any $x \in X$. So suppose that X does not contain such a vertex y. Consider a X-valid sequence of moves of the robber having a maximum number of steps before the capture of the robber. Let $u \in X$ be the position occupied by the cop before the capture of $\mathcal{R}$ and let $v \in X$ be the position of the robber at this step. Since wherever the robber moved next in X (including remaining in v or passing via u), the cop would capture him, necessarily $N_s(v, G \setminus \{u\}) \cap X \subseteq N_{s'}(u)$ holds. Set $X' := X \setminus \{v\}$.

We assert that X' is (s, s')-winnable as well. In this proof, we use a strategy that is not positional but uses one bit of memory. A strategy using one bit memory can be presented as follows: it is a function which takes as input the current positions of the two players and a boolean (the current value of the memory) and that outputs the next position of the cop and a boolean (the new value of the memory). Using the positional winning strategy σ, we define $\sigma'(c, r, m)$ for any positions $c \in X'$ of the cop and $r \in X'$ of the robber and for any value of the memory $m \in \{0, 1\}$. The intuitive idea for defining σ' is that the cop plays using σ except when he is in u and his memory contains 1; in this case, he uses σ as if he was in v. If $m = 0$ or $c \neq u$, then we distinguish two cases: if $\sigma(c, r) = v$ then $\sigma'(c, r, m) = (u, 1)$ (this is a valid move since $N_{s'}(v) \cap X \subseteq N_{s'}(u)$) and $\sigma'(c, r, m) = (\sigma(c, r), 0)$ otherwise. If $m = 1$ and $c = u$, we distinguish two cases: if $\sigma(v, r) = v$, then $\sigma'(u, r, 1) = (u, 1)$ and $\sigma'(u, r, 1) = (\sigma(v, r), 0)$ otherwise (this is a valid move since $N_{s'}(v) \cap X \subseteq N_{s'}(u)$). Let $S_r = (r_1, \ldots, r_p, \ldots)$ be any

X'-valid sequence of moves of the robber. Since $X' \subset X$, S_r is also a X-valid sequence of moves of the robber. Let $S_c := (c_1, \ldots, c_p, \ldots)$ be the corresponding X-valid sequence of moves of the cop following σ against S_r in X and let $S'_c = (c'_1, \ldots, c'_p, \ldots)$ be the X'-valid sequence of moves of the cop following σ' against S_r. Note that the sequences of moves S_c and S'_c differ only if $c_k = v$ and $c'_k = u$. Finally, since the cop follows a winning strategy for X, there is a step j such that $c_j = r_j \in X \setminus \{v\}$ (note that $r_j \neq v$ because we supposed that $S_r \subseteq X'$). Since $c_j \neq v$, we also have $c'_j = r_j$, thus $\mathcal{C}$ captures $\mathcal{R}$ in the X'-restricted game. Starting from a positional strategy for the X-restricted game, we have constructed a winning strategy using memory for the X'-restricted game. As mentioned in the introduction, it implies that there exists a positional winning strategy for the X'-restricted game.

Applying induction on the number of vertices of the cop-winning set X, we conclude that X is (s, s')-dismantlable. Applying this assertion to the vertex set V of cop-win graph $G = (V, E)$ from the class $\mathcal{CWFR}(s, s')$, we will conclude that G is (s, s')-dismantlable. □

Corollary 1. *Given a graph $G = (V, E)$ and the integers $s, s' \in \mathbb{N} \cup \{\infty\}$, $s' \leq s$, one can recognize in polynomial time if G belongs to $\mathcal{CWFR}(s, s')$.*

Proof. By Theorem 1, $G \in \mathcal{CWFR}(s, s')$ if and only if G is (s, s')-dismantlable. Moreover, from the last part of the proof of Theorem 1 we conclude that if a subset X of vertices of G is (s, s')-winnable and $u, v \in X$ such that $N_s(v, G \setminus \{u\}) \cap X \subseteq N_{s'}(u)$ holds, then the set $X' = X \setminus \{v\}$ is (s, s')-winnable as well. Therefore it suffices to run the following algorithm. Start with $X := V$ and as long as possible find in X two vertices u, v satisfying the inclusion $N_s(v, G \setminus \{u\}) \cap X \subseteq N_{s'}(u)$, and set $X := X \setminus \{v\}$. If the algorithm ends up with a set X containing at least two vertices, then G is not (s, s')-winnable, otherwise, if X contains a single vertex, then G is (s, s')-dismantlable and therefore $G \in \mathcal{CWFR}(s, s')$. □

2.2. **Graphs of $\mathcal{CWFR}(s, s')$ and hyperbolicity.** Introduced by Gromov [23], δ-hyperbolicity of a metric space measures, to some extent, the deviation of a metric from a tree metric. A graph G is *δ-hyperbolic* if for any four vertices u, v, x, y of G, the two larger of the three distance sums $d(u, v) + d(x, y)$, $d(u, x) + d(v, y)$, $d(u, y) + d(v, x)$ differ by at most $2\delta \geq 0$. Every 4-point metric d has a canonical representation in the rectilinear plane as illustrated in Fig. 1. The three distance sums are ordered from small to large, thus implying $\xi \leq \eta$. Then η is half the difference of the largest and the smallest sum, while ξ is half the largest minus the medium sum. Hence, a graph G is δ-hyperbolic iff ξ does not exceed δ for any four vertices u, v, w, x of G. Many classes of graphs are known to have bounded hyperbolicity [6, 16]. Our next result, based on Theorem 1 and a result of [16], establishes that in a δ-hyperbolic graph a "slow" cop captures a faster robber provided that $s' > s/2 + 2\delta$ (in the same vein, Benjamini [8] showed that in the competition of two growing clusters in a δ-hyperbolic graph, one growing faster that the other, the faster cluster not necessarily surround the slower cluster).

Proposition 1. *Given $r \geq 2\delta \geq 0$, any δ-hyperbolic graph $G = (V, E)$ is $(2r, r + 2\delta)$-dismantlable and therefore $G \in \mathcal{CWFR}(2r, r + 2\delta)$.*

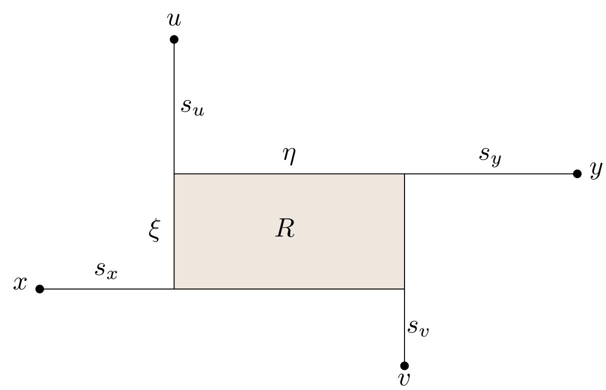


FIGURE 1. Realization of a 4-point metric in the rectilinear plane.

Proof. The second assertion follows from Theorem 1. To prove the $(2r, r+2\delta)$-dismantlability of G, we will employ Lemma 2 of [16]. According to this result, in a δ-hyperbolic graph G for any subset of vertices X there exist two vertices $x \in X$ and $c \in V$ such that $d(c, y) \le r + 2\delta$ for any vertex $y \in X \cap N_{2r}(x)$, i.e., $N_{2r}(x) \cap X \subseteq N_{r+2\delta}(c)$. The proof of [16] shows that the vertices x and c can be selected in the following way: pick any vertex z of G as a basepoint, construct a breadth-first search tree T of G rooted at z, and then pick x to be the furthest from z vertex of X and c to be vertex located at distance $r+2\delta$ from x on the unique path between x and z in T. Using this result, we will establish a slightly stronger version of dismantlability of a δ-hyperbolic graph G, in which the inclusion $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_{s'}(v_j)$ is replaced by $N_s(v_i) \cap X_i \subseteq N_{s'}(v_j)$ with $s := 2r$ and $s' := r + 2\delta$. We recursively construct the ordering of V. By previous result, there exist two vertices $v_1 \in X_1 := V$ and $c \in X_2 := V \setminus \{v_1\}$ such that $N_{2r}(v_1) \cap X_1 \subseteq N_{r+2\delta}(c)$. At step $i \ge 1$, suppose by induction hypothesis that V is the disjoint union of the sets $\{v_1, \ldots, v_i\}$ and X_{i+1}, so that, for any $j \le i$, there exists a vertex $c \in X_{j+1}$ such that $N_{2r}(v_j) \cap X_j \subseteq N_{r+2\delta}(c)$ with $X_j = \{v_j, \ldots, v_i\} \cup X_{i+1}$. We assert that this ordering can be extended. Applying the previous result to the set $X := X_{i+1}$ we can define two vertices $v_{i+1} \in X_{i+1}$ and $c \neq v_{i+1}$ such that $N_{2r}(v_{i+1}) \cap X_{i+1} \subseteq N_{r+2\delta}(c)$. The choice of the vertices $x \in X$ and $c \in V$ provided by [16] and the definition of the sets $X_1, X_2, \ldots$ ensure that if a vertex of G is closer to the root than another vertex, then the first vertex will be labeled later than the second one. Since by construction c is closer to z than v_{i+1}, necessarily c belongs to the set $X_{i+1} \setminus \{v_{i+1}\}$. $\square$

In general, dismantlable graphs do not have bounded hyperbolicity because they are universal in the following sense. As we noticed in the introduction, any finite Helly graph is dismantlable. On the other hand, it is well known that an arbitrary connected graph can be isometrically embedded into a Helly graph (see for example [6, 31]). However, dismantlable graphs without some short induced cycles are 1-hyperbolic:

Corollary 2. *Any dismantlable graph $G = (V, E)$ without induced 4-,5-, and 6-cycles is 1-hyperbolic, and therefore $G \in \mathcal{CWFR}(2r, r+2)$ for any $r > 0$.*

Proof. A dismantlable graph G not containing induced 4- and 5-cycles does not contain 4-wheels and 5-wheels as well (a *k-wheel* is a cycle of length k plus a vertex adjacent to all vertices of this cycle), therefore G is bridged by a result of [1]. Since G does not contain 6-wheels as well, G is 1-hyperbolic by Proposition 11 of [16]. Then the second assertion immediately follows from Proposition 1. $\square$

Open question 1: Is it true that the converse of Proposition 1 holds? More precisely, is it true that if $G \in \mathcal{CWFR}(s,s')$ for $s' < s$, then the graph G is δ-hyperbolic, where δ depends only of s?

We give some confidences in the truth of this conjecture by showing that for $s \geq 2s'$ all graphs $G \in \mathcal{CWFR}(s,s')$ are $(s-1)$-hyperbolic. On the other hand, since $\mathcal{CWFR}(s,s') \subset \mathcal{CWFR}(s,s'+1)$, to answer our question for $s' < s < 2s'$ it suffices to show its truth for the particular case $s' = s-1$. We give a positive answer to our question for Helly and bridged graphs by showing that if such a graph G belongs to the class $\mathcal{CWFR}(s,s-1)$, then G is s^2-hyperbolic.

In the following results, for an (s,s')-dismantling order $v_1,\ldots,v_n$ of a graph $G \in \mathcal{CWFR}(s,s')$ and a vertex v of G, we will denote by $\alpha(v)$ the rank of v in this order (i.e., $\alpha(v) = i$ if $v = v_i$). For two vertices u,v with $\alpha(u) < \alpha(v)$ and a shortest (u,v)-path $P(u,v)$, an *s-net* $N(u,v)$ of $P(u,v)$ is an ordered subset $(u = x_0, x_1, \ldots, x_k, x_{k+1} = v)$ of vertices of $P(u,v)$, such that $d(x_i,x_{i+1}) = s$ for any $i = 0,\ldots,k-1$ and $0 < d(x_k,x_{k+1}) \leq s$.

Proposition 2. *If $G \in \mathcal{CWFR}(s,s-1)$ and u,v are two vertices of G such that $\alpha(u) < \alpha(v)$ and $d(u,v) > s^2$, then for any shortest (u,v)-path $P(u,v)$, the vertex x_1 of its s-net $N(u,v) = (u = x_0, x_1, \ldots, x_k, x_{k+1} = v)$ satisfies the condition $\alpha(u) < \alpha(x_1)$.*

Proof. Suppose by way of contradiction that $\alpha(u) > \alpha(x_1)$. Let x_i $(1 \leq i \leq k)$ be a vertex of $N(u,v)$ having a locally minimal index $\alpha(x_i)$, i.e., $\alpha(x_{i-1}) > \alpha(x_i) < \alpha(x_{i+1})$. Let y_i be the vertex eliminating x_i in the $(s,s-1)$-dominating order. We assert that $d(y_i,x_{i-1}) \leq s-1$ and $d(y_i,x_{i+1}) \leq s-1$. Indeed, if y_i does not belong to the portion of the path $P(u,v)$ comprised between x_{i-1} and x_{i+1}, then $x_{i-1}, x_{i+1} \in X_{\alpha(x_i)} \cap N_s(x_i, G \setminus \{y_i\})$, and therefore $x_{i-1}, x_{i+1} \in N_{s-1}(y_i)$ by the dismantling condition. Now suppose that y_i belongs to one of the segments of $P(u,v)$, say to the subpath between x_{i-1}, x_i. Since $y_i \neq x_i$ we conclude that $d(x_{i-1},y_i) \leq s-1$. On the other hand, since $x_{i+1} \in X_{\alpha(x_i)} \cap N_s(x_i, G \setminus \{y_i\})$, by dismantling condition we conclude that $d(y_i,x_{i+1}) \leq s-1$. Hence, indeed $d(y_i,x_{i-1}) \leq s-1, d(y_i,x_{i+1}) \leq s-1$, whence $d(x_{i-1},x_{i+1}) \leq 2s-2$. Since $d(x_{i-1},x_{i+1}) = 2s$ for any $1 \leq i \leq k-1$, we conclude that $i = k$. Therefore the indices of the vertices of $N(u,v)$ satisfy the inequalities $\alpha(u) = \alpha(x_0) > \ldots > \alpha(x_{k-1}) > \alpha(x_k) < \alpha(x_{k+1}) = \alpha(v)$.

Denote by N the ordered sequence of vertices $x_0 = u, x_1, \ldots, x_{k-1}, y_k, x_{k+1} = v$ obtained from the s-net $N(u,v)$ by replacing the vertex x_k by y_k. We say that N is obtained from $N(u,v)$ by an *exchange*. Call two consecutive vertices of N a *link*; N has $k+1$ links, namely, $k-1$ links of length s and two links of length at most $s-1$. If $\alpha(y_k) < \alpha(x_{k-1})$, then we perform with y_k the same exchange operation as we did with x_k. After several such

exchanges, we will obtain a new ordered set $x_0 = u, x_1, \ldots, x_{k-1}, z_k, x_{k+1} = v$ (denote it also by N) having $k-1$ links of length s and two links of length $\leq s-1$ and $\alpha(x_{k-1}) < \alpha(z_k)$. Since $\alpha(x_{k-2}) > \alpha(x_{k-1})$, using the $(s, s-1)$-dismantling order we can exchange in N the vertex x_{k-1} by a vertex y_{k-1} to get an ordered set (denote it also by N) having $k-3$ links of length s and 3 links of length $s-1$. Repeating the exchange operation with each occurring local minimum (different from u) of N with respect to the total order α, after a finite number of exchanges we will obtain an ordered set $N = (u, z_1, z_2, \ldots, z_k, v)$ consisting of $k+1$ links of length at most $s-1$ each and such that $\alpha(u) < \alpha(z_i)$ for any $i = 1, \ldots, k$. By triangle inequality, $d(u,v) \leq d(u,z_1) + d(z_1,z_2) + \ldots + d(z_k,v) \leq (k+1)(s-1)$. On the other hand, from the definition of $N(u,v)$ we conclude that $d(u,v) = ks + \gamma$, where $0 < \gamma = d(x_k, v) \leq s$. Hence $(k+1)(s-1) \geq ks + \gamma$, yielding $k \leq s - \gamma - 1$. But then $d(u,v) = ks + \gamma \leq (s-\gamma-1)s + \gamma = s^2 - s\gamma - s + \gamma < s^2$, contrary to the assumption that $d(u,v) \geq s^2$. This contradiction shows that indeed $\alpha(x_1) > \alpha(u)$. $\square$

We call a graph $G \in \mathcal{CWFR}(s, s-1)$ $(s, s-1)^*$-*dismantlable* if for any $(s, s-1)$-dismantling order $v_1, \ldots v_n$ of G, for each vertex $v_i, 1 \leq i < n$, there exists another vertex v_j *adjacent* to v_i such that $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_{s-1}(v_j)$, where $X_i := \{v_i, v_{i+1}, \ldots, v_n\}$ and $X_n = \{v_n\}$. The difference between $(s, s-1)$-dismantlability and $(s, s-1)^*$-dismantlability is that in the second case the vertex v_j dominating v_i is necessarily adjacent to v_i but not necessarily eliminated after v_i.

Proposition 3. *If a graph $G \in \mathcal{CWFR}(s, s-1)$ is $(s, s-1)^*$-dismantlable, then G is s^2-hyperbolic.*

Proof. Pick any quadruplet of vertices u, v, x, y of G, consider its representation as in Fig. 1 where $\xi \leq \eta$, and proceed by induction on the total distance sum $S(u,v,x,y) = d(u,v) + d(u,x) + d(u,y) + d(v,x) + d(v,y) + d(x,y)$. From Fig. 1 we immediately conclude that if one of the distances between the vertices u, v, x, y is at most s^2, then $\xi \leq s^2$ and we are done. So suppose that the distance between any two vertices of our quadruplet is at least s^2.

Consider any $(s, s-1)$-dismantling order $v_1, \ldots, v_n$ of G and suppose that u is the vertex of our quadruplet occurring first in this order. Pick three shortest paths $P(u,v), P(u,x)$, and $P(u,y)$ between the vertex u and the three other vertices of the quadruplet. Denote by v_1, x_1, and y_1 the vertices of the paths $P(u,v), P(u,x)$, and $P(u,y)$, respectively, located at distance s from u. From Proposition 2 we infer that u is eliminated before each of the vertices v_1, x_1, y_1. Let u' be the neighbor of u eliminating u in the $(s, s-1)^*$-dismantling order associated with the $(s, s-1)$-dismantling order $v_1, \ldots, v_n$. From the $(s, s-1)^*$-dismantling condition we infer that each of the distances $d(u', v_1), d(u', x_1), d(u', y_1)$ is at most $s-1$. Since u is adjacent to u' and u is at distance s from v_1, x_1, y_1, necessarily $d(u', v_1), d(u', x_1), d(u', y_1)$ are all equal to $s-1$. Therefore, if we will replace in our quadruplet the vertex u by u', we will obtain a quadruplet with a smaller total distance sum: $S(u', v, x, y) = S(u, v, x, y) - 3$. Therefore, by induction hypothesis, the two largest of the distance sums $d(u', v) + d(x, y)$, $d(u', x) + d(v, y)$, $d(u', y) + d(v, x)$ differ by at most $2s^2$. On the other hand, $d(u,v) + d(x,y) = d(u',v) + d(x,y) + 1$, $d(u,x) + d(v,y) = d(u',x) + d(v,y) + 1$, and $d(u,y) + d(v,x) = d(u',y) + d(v,x) + 1$, whence

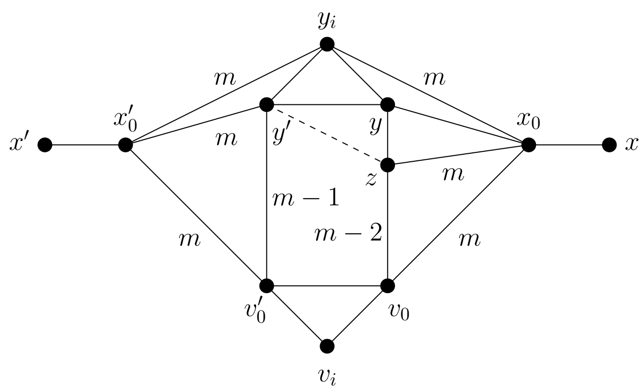


FIGURE 2. To the proof of Proposition 4 (case of bridged graphs).

the two largest distance sums of the quadruplet u, v, x, y also differ by at most $2s^2$. Hence G is s^2-hyperbolic. $\square$

A graph G is called a *Helly graph* if its family of balls satisfies the Helly property: any collection of pairwise intersecting balls has a common vertex. A graph G is called a *bridged graph* if all isometric cycles of G have length 3. Equivalently, G is a bridged graph if all balls around convex sets are convex (a subset S of vertices is convex if together with any two vertices u, v, the set S contains the *interval* $I(u,v) = \{x \in V : d(u,v) = d(u,x) + d(x,v)\}$ between u and v). For a comprehensive survey of results and bibliography on Helly and bridged graphs, see [6].

Proposition 4. *If $G \in \mathcal{CWFR}(s, s-1)$ is a Helly or a bridged graph, then G is $(s, s-1)^*$-dismantlable and therefore G is s^2-hyperbolic.*

Proof. The second assertion immediately follows from Proposition 3. Thus, we only need to prove that any Helly or bridged graph in $\mathcal{CWFR}(s, s-1)$ is $(s, s-1)^*$-dismantlable.

First, let G be an $(s, s-1)$-dismantlable Helly graph. Let v_i be the ith vertex in an $(s, s-1)$-dismantling order and let y_i be the vertex eliminating v_i. Suppose that $k := d(v_i, y_i) \geq 2$. We assert that we can always eliminate v_i with a vertex y_i' adjacent to y_i and located at distance $k-1$ from v_i. Then repeating the same reasoning with y_i' instead of y_i, we will eventually arrive at a vertex of $I(v_i, y_i)$ adjacent to v_i which still eliminates v_i. Set $A := (X_i \cap N_s(v_i)) \setminus \{v_i, y_i\}$. For each vertex $x \in A$, consider the ball $N_{s-1}(x)$ of radius $s-1$ centered at x. Consider also the balls $N_{k-1}(v_i)$ and $N_1(y_i)$. We assert that the balls of the resulting collection pairwise intersect. Indeed, any two balls centered at vertices of A intersect in y_i. The ball $N_1(y_i)$ intersects any ball centered at A in y_i. The ball $N_{k-1}(v_i)$ intersects any ball centered at a vertex $x \in A$ because $d(v_i, x) \leq s \leq k-1+s-1$. Finally, $N_{k-1}(v_i)$ and $N_1(y_i)$ intersect because $d(v_i, y_i) = k = k-1+1$. By Helly property, the balls of this collection intersect in a vertex y_i'. Since y_i' is at distance at most $k-1$ from v_i and at distance at most 1 from y_i, from the equality $d(v_i, y_i) = k$ we immediately deduce that y_i' is a neighbor of y_i located at

distance $k-1$ from v_i. This establishes the $(s,s-1)^*$-dismantling property for Helly graphs in $\mathcal{CWFR}(s,s-1)$.

Now, suppose that G is a bridged graph and let the vertices v_i, y_i and the set A be defined as in the previous case. Since G is bridged, the convexity of the ball $N_{k-1}(v_i)$ implies that the set C of neighbors of y_i in the interval $I(v_i,y_i)$ induces a complete subgraph. Pick any vertex $x \in A$. Clearly, $d(x,y_i) \leq s-1$ and $d(x,v_i) \leq s$. If $d(x,v_i) \leq s-1$, then $v_i, y_i \in N_{s-1}(x)$ and from the convexity of the ball $N_{s-1}(x)$ we conclude that $I(v_i,y_i) \subset N_{s-1}(x)$. Hence, in this case, $d(x,y) \leq s-1$ for any $y \in I(v_i,y_i)$, in particular, for any vertex of C. Analogously, if $d(x,y_i) < s-1$, then $d(x,y) \leq s-1$ for any vertex $y \in C$. Therefore the choice of the vertex y_i' in C depends only of the vertices of the set $A_0 = \{x \in A : d(x,v_i) = s \text{ and } d(x,y_i) = s-1\}$.

Pick any vertex $x \in A_0$. If $I(x,y_i) \cap I(y_i,v_i) \neq \{y_i\}$, then y_i has a neighbor y' in this intersection located at distance $s-2$ from x. Since $y' \in C$ and C is a complete subgraph, then $d(y,x) \leq s-1$ for any $y \in C$. Therefore we can discard all such vertices of A_0 from our future analysis and suppose without loss of generality that $I(x,y_i) \cap I(y_i,v_i) = \{y_i\}$ for any $x \in A_0$. For $x \in A_0$, let x_0 be a furthest from x vertex of $I(x,y_i) \cap I(x,v_i)$. Let v_0 be a furthest from v_i vertex of $I(v_i,x_0) \cap I(v_i,y_i)$. Since $I(x,y_i) \cap I(y_i,v_i) = \{y_i\}$ and G is bridged, the vertices y_i, x_0, v_0 define an equilateral metric triangle sensu [5, 6]: $d(y_i,x_0) = d(x_0,v_0) = d(v_0,y_i) =: m$. Moreover, any vertex of $I(v_0,y_i)$ is located at distance m from x_0 and therefore at distance $s-1$ from x, showing, in particular, that $N_{s-1}(x) \cap C \neq \emptyset$ for any $x \in A_0$. From the definition of x_0 and v_0 we conclude that $m + d(x_0,x) = s-1, d(x,x_0) + m + d(v_0,v_i) = s$, and $d(v_i,v_0) + m \leq s-1$. Whence $d(v_i,v_0) = 1$, yielding $d(v_i,y_i) = m+1$.

Pick in C a vertex y belonging to a maximum number of balls $N_{s-1}(x)$ centered at $x \in A_0$. Suppose by way of contradiction that A_0 contains a vertex x' such that $y \notin N_{s-1}(x')$ (for an illustration, see Fig. 2). Since $d(x',y_i) = s-1$ and y is adjacent to y_i, we have $d(x',y) = s$. Let y' be a vertex of C belonging to $N_{s-1}(x')$ (such a vertex y' exists because of the remark in above paragraph). Let v_0' be the neighbor of v_i defined with respect to x' in the same way as v_0 was defined for x. Then all vertices of $I(v_0',y')$ are located at distance $s-1$ from x'. We can suppose that there exists a vertex $x \in A_0$ such that $y \in N_{s-1}(x)$ but $y' \notin N_{s-1}(x)$, otherwise we will obtain a contradiction with the choice of y. Since the balls $N_{s-1}(x)$ and $N_{s-1}(x')$ are convex, the intervals $I(v_0,y_i)$ and $I(v_0',y_i)$ belong to these balls, respectively, whence $d(v_0,y) = d(v_0',y') = m-1$ but $d(v_0,y') = d(v_0',y) = m$. Let z be a neighbor of y in $I(v_0,y)$. Since $z, y' \in I(y,v_0')$ and G is bridged, the vertices z and y' are adjacent. Hence $y' \in I(v_0,y_i)$, yielding $d(x,y') = s-1$, contrary to our assumption that $y' \notin N_{s-1}(x)$. This contradiction shows that C contains a vertex belonging to all balls $N_{s-1}(x)$ centered at vertices of A_0, thus establishing the $(s,s-1)^*$-dismantling property for bridged graphs in $\mathcal{CWFR}(s,s-1)$. $\square$

Proposition 5. *If $s \geq 2s'$, then any graph G of $\mathcal{CWFR}(s,s')$ is $(s-1)$-hyperbolic.*

Proof. First, similarly to Proposition 2, we prove that if $d(u,v) \geq s$ and $\alpha(u) < \alpha(v)$, then the vertex x_1 of the s-net $N(u,v)$ of any shortest (u,v)-path satisfies the inequality $\alpha(x_1) > \alpha(u)$. Suppose by way of contradiction that $\alpha(u) > \alpha(x_1)$. Then as in proof of Proposition 2 we conclude that x_k is the unique local minimum of α on $N(u,v) : \alpha(x_{k-1}) > \alpha(x_k) < \alpha(x_{k+1})$.

Let y_k be the vertex eliminating x_k in the (s,s')-dominating order. If y_k does not belong to the segment of $P(u,v)$ between x_{k-1} and x_k, then $d(x_{k-1},x_{k+1}) \le d(x_{k-1},y_k)+d(y_k,x_{k+1}) \le 2s'$, contrary to the assumption that $d(x_{k-1},x_{k+1}) > s \ge 2s'$. So y_k belongs to the subpath of $P(u,v)$ between x_{k-1} and x_{k+1}. If y_k belongs to the subpath comprised between x_k and x_{k+1}, then the dismantling condition implies that $d(y_k,x_{k-1}) \le s'$, which is impossible because $d(y_k,x_{k-1}) = d(y_k,x_k) + s > 2s'$. The same contradiction is obtained if y_k belongs to the second half of the subpath between x_{k-1} and x_k. Finally, if y_k belongs to the first half of this subpath, then $d(y_k,x_{k+1}) \le s'$ by the dismantling condition, contradicting the fact that the location of y_k on this subpath of $P(u,v)$ implies that $d(y_k,x_{k+1}) > s'$. This shows that indeed $\alpha(x_1) > \alpha(u)$.

To establish $(s-1)$-hyperbolicity of G, as in the proof of Proposition 3 we pick any quadruplet of vertices u,v,x,y of G and proceed by induction on the total distance sum $S(u,v,x,y) = d(u,v)+d(u,x)+d(u,y)+d(v,x)+d(v,y)+d(x,y)$. Again, we can suppose that the distances between any two vertices of this quadruplet is at least s, otherwise we are done. Let u be the vertex of our quadruplet occurring first in some (s,s')-dismantling order of G. Pick three shortest paths $P(u,v), P(u,x)$, and $P(u,y)$ and denote by v_1, x_1, and y_1 their respective vertices located at distance s from u. From first part of our proof we infer that u is eliminated before v_1, x_1, and y_1. Let u' be the vertex eliminating u. From the (s,s')-dismantling condition we infer that $d(u,u') \le s'$. Moreover, either $d(u',v_1) \le s'$ or $v_1 \notin N_s(u, G \setminus \{u'\})$. Since $d(u,v_1) = s \ge 2s'$, in both cases we conclude that u' belongs to a shortest (u,v_1)-path of G. Analogously, we conclude that u' lie on a shortest (u,x_1)-path and on a shortest (u,y_1)-path. Therefore, if we replace in our quadruplet u by u', we will get a quadruplet with total distance sum $S(u',v,x,y) = S(u,v,x,y) - 3d(u,u') < S(u,v,x,y)$. By induction hypothesis, the two largest distance sums of this quadruplet differ by at most $2(s-1)$. On the other hand, since $d(u,v)+d(x,y) = d(u',v)+d(x,y)+d(u,u')$, $d(u,x)+d(v,y) = d(u',x)+d(v,y)+d(u,u')$, and $d(u,y)+d(v,x) = d(u',y)+d(v,x)+d(u,u')$, the two largest distance sums of the quadruplet u,v,x,y also differ by at most $2(s-1)$. Hence G is $(s-1)$-hyperbolic. $\square$

3. Cop-win graphs for game with fast robber: class $\mathcal{CWFR}(s)$

In this section, we specify the dismantling scheme provided by Theorem 1 in order to characterize the graphs in which one cop with speed 1 captures a robber with speed $s \ge 2$. First we show that the graphs from $\mathcal{CWFR}(2)$ are precisely the dually chordal graphs [12]. Then we show that for $s \ge 3$ the classes $\mathcal{CWFR}(s)$ coincide with $\mathcal{CWFR}(\infty)$ and we provide a structural characterization of these graphs.

3.1. $\mathcal{CWFR}(2)$ **and dually chordal graphs.** We start by showing that when the cop has speed 1 and the robber has speed $s \ge 1$, then the dismantling order in Theorem 1 can be defined using the subgraphs $G_i = G(X_i)$.

Proposition 6. *A graph G is $(s,1)$-dismantlable if and only if the vertices of G can be ordered $v_1, \dots, v_n$ in such a way that for each vertex $v_i \neq v_n$ there exists a vertex v_j with $j > i$ such that $N_s(v_i, G_i \setminus \{v_j\})) \subseteq N_1(v_j, G_i)$.*

Proof. First, note that for any $i \leq j$, $N_1(v_j, G) \cap X_i = N_1(v_j, G_i)$. Thus, if a graph G is $(s,1)$-dismantlable, then any $(s,1)$-dismantling order satisfies the requirement $N_s(v_i, G_i \setminus \{v_j\})) \subseteq N_1(v_j, G_i)$. Conversely, consider an order $v_1, \dots, v_n$ on the vertices of G satisfying this condition. If $s = 1$, then $N_1(v_i, G_i \setminus \{v_j\})) = N_1(v_i, G \setminus \{v_j\}) \cap X_i$ and thus our assertion is obviously true. We now suppose that $s \geq 2$. By induction on i, we will show that $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_1(v_j)$. For $i = 1$, $G_i = G$ and thus the property holds. Consider i such that for any $i' < i$, the property is satisfied. Pick any vertex $u \in N_s(v_i) \cap X_i$. If the distance in $G_i \setminus \{v_j\}$ between v_i and u is at most s, then $u \in N_s(v_i, G_i \setminus \{v_j\}) \subseteq N_1(v_j)$ and we are done. Otherwise, we can find a unique index $i_0 < i$ such that the distance between v_i and u in the graph $G_{i_0} \setminus \{v_j\}$ is at most s and in the graph $G_{i_0+1} \setminus \{v_j\}$ is larger than s. Consider a shortest path π between v_i and u in $G_{i_0} \setminus \{v_j\}$. From the choice of i_0, necessarily v_{i_0} is a vertex of π. Since the length of π is at most s, we deduce that $d_{G_{i_0}}(u, v_{i_0}) \leq s$ and $d_{G_{i_0}}(v_i, v_{i_0}) \leq s$. By the induction hypothesis, there exists $j_0 > i_0$ such that $N_s(v_{i_0}, G_{i_0} \setminus \{v_{j_0}\})) \subseteq N_1(v_{j_0})$. If $j_0 \neq j$, then there exists a path (u, v_{j_0}, v_i) of length 2 between u and v_i in G_{j_0}. Since $j_0 > i_0$, we obtain a contradiction with the definition of i_0. Hence $j_0 = j$, and, by our induction hypothesis, $u \in N_s(v_{i_0}, G_{i_0} \setminus \{v_j\})) \subseteq N_1(v_j)$, and we are done. $\square$

Analogously to Theorem 3 of Clarke [18] for the witness version of the game, it can be easily shown that, for any s, the class $\mathcal{CWFR}(s)$ is closed under retracts:

Proposition 7. *If $G \in \mathcal{CWFR}(s)$ and G' is a retract of G, then $G' \in \mathcal{CWFR}(s)$.*

Recall that a graph G is called *dually chordal* [12] if its clique hypergraph (or, equivalently, its ball hypergraph) is a hypertree, i.e., it satisfies the Helly property and its line graph is chordal (see the Berge's book on hypergraphs [10] for these two definitions). Dually chordal graphs are equivalently defined as the graphs G having a spanning tree T such that any maximal clique or any ball of G induces a subtree of T. Finally, dually chordal graphs are exactly the graphs $G = (V, E)$ admitting a maximum neighborhood ordering of its vertices. A vertex $u \in N_1(v)$ is a *maximum neighbor* of v if for all $w \in N_1(v)$ the inclusion $N_1(w) \subseteq N_1(u)$ holds. The ordering $\{v_1, \dots, v_n\}$ is a *maximum neighborhood ordering* (*mno* for short) of G [12], if for all $i < n$, the vertex v_i has a maximum neighbor in the subgraph G_i induced by the vertices $X_i = \{v_i, v_{i+1}, \dots, v_n\}$. Dually chordal graphs comprise strongly chordal graphs, doubly chordal, and interval graphs as subclasses and can be recognized in linear time. Any graph H can be transformed into a dually chordal graph by adding a new vertex c adjacent to all vertices of H.

Theorem 2. *For a graph $G = (V, E)$, the following conditions are equivalent:*

(i) *$G \in \mathcal{CWFR}(2)$;*
(ii) *G is $(2,1)$-dismantlable;*
(iii) *G admits an mno ordering;*

(iv) G *is dually chordal.*

Proof. Since $\mathcal{CWFR}(2) = \mathcal{CWFR}(2,1)$, the equivalence (i)⇔(ii) follows from Theorem 1. The equivalence (iii)⇔(iv) is a result of [12]. Notice that u is a maximum neighbor of v in G iff $N_2(v) = N_1(u)$. Therefore, $\{v_1, \ldots, v_n\}$ is a maximum neighborhood ordering of G iff for all $i < n$, $N_2(v_i, G_i) = N_1(v_j, G_i)$ for some $v_j, j > i$. Hence any mno ordering is a $(2,1)$-dismantling ordering, establishing (iii)⇒(ii). Finally, by induction on the number of vertices of G we will show that any $(2,1)$-dismantling ordering $\{v_1, \ldots, v_n\}$ of the vertex set of G is an mno, thus (ii)⇒(iii). Suppose that $N_2(v_1, G \setminus \{u\}) \subset N_1(u)$ for some $u := v_j, j > 1$. Then u is adjacent to v_1 and to all neighbors of v_1. Since for any neighbor $w \neq u$ of v_1 the ball $N_1(w)$ is contained in the punctured ball $N_2(v_1, G \setminus \{u\})$, we conclude that $N_1(w) \subseteq N_1(u)$, i.e., u is a maximum neighbor of v_1. The graph G' obtained from G by removing the vertex v_1 is a retract, and therefore an isometric subgraph of G. Thus for any vertex $v_i, i > 1$, by what has been noticed above (Proposition 6), the intersection of a ball (or of a punctured ball) of G centered at v_i with the set $X_2 = \{v_2, \ldots, v_n\}$ coincides with the corresponding ball (or punctured ball) of the graph $G' = G(X_2)$ centered at the same vertex v_i. Therefore $\{v_2, \ldots, v_n\}$ is a $(2,1)$-dismantling ordering of the graph G'. By induction assumption, $\{v_2, \ldots, v_n\}$ is an mno of G'. Since v_1 has a maximum neighbor in $\{v_2, \ldots, v_n\}$, we conclude that $\{v_1, v_2, \ldots v_n\}$ is a maximum neighborhood ordering of G. □

3.2. $\mathcal{CWFR}(k), k \geq 3$, **and big brother graphs.** A *block* of a graph G is a maximal by inclusion vertex two-connected subgraph of G (possibly reduced to a single edge). Two blocks of G are either disjoint or share a single vertex, called an *articulation point*. Any graph $G = (V, E)$ admits a block-decomposition in the form of a rooted tree T: each vertex of T is a block of G, pick any block B_1 as a root of T, label it, and make it adjacent in T to all blocks intersecting it, then label that blocks and make them adjacent to all nonlabeled blocks which intersect them, etc. A block B of G is *dominated* if it contains a vertex u (called the *big brother* of B) which is adjacent to all vertices of B. A graph G is a *big brother graph*, if its block-decomposition can be represented in the form of a rooted tree T is such a way that (1) each block of G is dominated and (2) for each block B distinct from the root B_1, the articulation point between B and its father-block dominates B. Equivalently, G is a big brother graph if its blocks can be ordered $B_1, \ldots, B_r$ such that B_1 is dominated and, for any $i > 1$, the block B_i is a leaf in the block-decomposition of $\cup_{j \leq i} B_j$ and is dominated by the articulation point connecting B_i to $\cup_{j<i} B_j$ (we will call such a decomposition a *bb-decomposition* of G); see Fig. 3(a) for an example.

Theorem 3. *For a graph $G = (V, E)$ the following conditions are equivalent:*

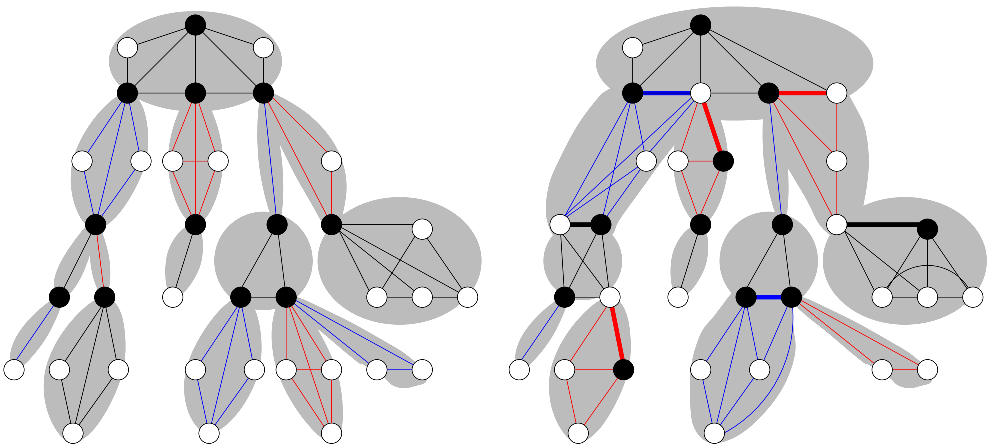
(i) $G \in \mathcal{CWFR}(3)$*;*

(i′) G *is* $(3,1)$*-dismantlable;*

(ii) $G \in \mathcal{CWFR}(\infty)$;

(ii′) G *is* $(\infty,1)$*-dismantlable;*

(iii) G *is a big brother graph.*



(a) (b)

FIGURE 3. (a) A big brother graph. (b) A big two-brother graph.

In particular, the classes of graphs $\mathcal{CWFR}(s), s \geq 3$, coincide.

Proof. The equivalences (i)$\Leftrightarrow$(i′) and (ii)$\Leftrightarrow$(ii′) are particular cases of Theorem 1. Next we will establish (iii)$\Rightarrow$(i)&(ii), i.e., that any big brother graph G belongs to $\mathcal{CWFR}(s)$ for all $s \geq 3$. Let $B_1, \ldots, B_r$ be a bb-decomposition of G. We consider the following strategy for the cop. At the beginning of the game, we locate the cop at the big brother of the root-block B_1. Now, at each subsequent step, the cop moves to the neighbor of his current position that is closest to the position of the robber. Notice the following invariant of the strategy: the position of the cop will always be at the articulation point of a block B on the path of T between the previous block hosting $\mathcal{C}$ and the current block hosting $\mathcal{R}$. This means that, since $\mathcal{R}$ cannot traverse this articulation point without being captured, $\mathcal{R}$ is restricted to move only in the union of blocks in the subtree rooted at B. Now, if before the move of the cop, $\mathcal{C}$ and $\mathcal{R}$ occupy their positions in the same block, then $\mathcal{C}$ captures $\mathcal{R}$ at the next move. Otherwise, the next move will increase the distance in T between the root and the block hosting $\mathcal{C}$. Therefore after at most diameter of T rounds, $\mathcal{R}$ and $\mathcal{C}$ will be located in the same block, and thus the cop captures the robber at next move. This shows that (iii)$\Rightarrow$(i)&(ii).

The remaining part of the proof is devoted to the implication (i)&(i′)$\Rightarrow$(iii). Let G be a graph of $\mathcal{CWFR}(3)$. Notice first that for any articulation point u of G, and any connected component C of $G \setminus \{u\}$, the graph induced by $C \cup \{u\}$ also belongs to $\mathcal{CWFR}(3)$. Indeed, this follows by noticing that $G(C \cup \{u\})$ is a retract of G (this retraction is obtained by mapping all vertices outside C to u) and that $\mathcal{CWFR}(3)$ is closed under retracts by Proposition 7. To prove that a graph $G = (V, E) \in \mathcal{CWFR}(3)$ is a big brother graph, we will proceed by induction on the number of vertices of G. If G has one or two vertices, the result is obviously true. For the inductive step, we distinguish two cases, depending if G is two-connected or not.

Case 1: G is not two-connected.

Since each block of G has strictly less vertices than G, by induction hypothesis each block is a big brother graph, i.e., it has a dominating vertex. First suppose that the block-decomposition of G has a leaf B such that the articulation point a of B separating B from the rest of G is a big brother of B. Let G' be the subgraph of G induced by all blocks of G except B, i.e., $G' = G(V \setminus (B \setminus \{a\}))$. Since $G' \in \mathcal{CWFR}(3)$ by what has been shown above, from the induction hypothesis we infer that G' is a big brother graph. Consequently, there exists a bb-decomposition $B_1, \ldots, B_r$ of G'. Then, $B_1, \ldots, B_r, B$ is a bb-decomposition of G and thus, G is a big brother graph. Suppose now that for any leaf in the block-decomposition of G, the articulation point of the corresponding block does not dominate it. Pick two leaves B_1 and B_2 in the block-decomposition of G and consider their unique articulation points a_1 and a_2 (a_i disconnects B_i from the rest of G). We claim that in this case, a robber that moves at speed 3 can always escape, which will contradicts the assumption that $G \in \mathcal{CWFR}(3)$. Let b_i be the dominating vertex of the block B_i, $i = 1, 2$ (by assumption, $b_i \neq a_i$). Consider now a vertex $c_i \in B_i \setminus \{b_i\}$ which can be connected with a_i by a 2-path (c_i, g_i, a_i) avoiding b_i (such a vertex exists because B_i is two-connected and, by assumption, a_i is not a dominating vertex of B_i). Let π be a shortest path from a_1 to a_2 in G and let h_1 and h_2 be the neighbors in π of a_1 and a_2, respectively. Note that h_i does not belong to B_i, thus a_i is the only neighbor of h_i in B_i. We now describe a strategy that enables the robber to escape. Initially, if the cop is not in B_1, then the robber starts in c_1; otherwise, he starts in c_2. Then the robber stays in c_i, as long as the cop is at distance ≥ 2 from c_i. When the cop moves to a neighboring vertex of c_i, then the robber goes to h_i (either via the path (c_i, b_i, a_i, h_i) or via the path (c_i, g_i, a_i, h_i)) and then, no matter how the cop moves, he goes to c_{3-i} using the shortest path π. Now notice that when $\mathcal{R}$ is in h_i, $\mathcal{C}$ is in $B_i \setminus \{a_i\}$ and thus he cannot capture the robber. When the robber is moving from h_i to c_{3-i}, he uses a shortest path π of G: the cop cannot capture him either because he is initially at distance 2 from the robber and he moves slower than the robber. Consequently, the cop cannot capture the robber, contrary with the assumption $G \in \mathcal{CWFR}(3)$.

Case 2: G is two-connected.

We must show that G has a dominating vertex. Consider a $(3, 1)$-dismantling order $v_1, \ldots, v_n$ of the vertices of G. Let u be a vertex such that $N_3(v_1, G \setminus \{u\}) \subseteq N_1(u)$. Since u is a maximum neighbor of v_1, the isometric subgraph $G' := G(V \setminus \{v_1\})$ of G also belongs to $\mathcal{CWFR}(3)$ because $v_2, \ldots, v_n$ is a $(3, 1)$-dismantling ordering of G'. By induction hypothesis, G' is a big brother graph. Again, we distinguish two subcases, depending on the two-connectivity of G'. First suppose that G' is two-connected. Since G' is a big brother graph, it contains a dominating vertex t. If t is adjacent to v_1, then t dominates G and we are done. Otherwise, consider a neighbor $w \neq u$ of v_1. Any vertex $x \neq u$ of G can be connected to v_1 by the path (v_1, w, t, x) of length 3 avoiding u, thus x belongs to the punctured ball $N_3(v_1, G \setminus \{u\})$. As a consequence, x is a neighbor of u, thus u dominates G. Now suppose that G' is not two-connected. We assert that u is the only articulation point of G'. Assume by way of contradiction that $w \neq u$ is an articulation point of G' and let x and y be two vertices

of G' such that all paths connecting x to y go through w. In G, x and y can be connected by two vertex-disjoint paths π_1 and π_2. Assume without loss of generality that $w \notin \pi_1$. Since π_1 cannot be a path of G', the vertex v_1 belongs to π_1. Let $\pi_1 = (x, x_1, \ldots, x_k, v_1, y_l, \ldots, y_1)$. Since $x_k, y_l \in N_1(v_1) \subseteq N_3(v_1, G \setminus \{u\}) \cup \{u\} \subseteq N_1(u)$, necessarily $x_k, y_l \in N_1(u)$. If $x_k = u$ or $y_l = u$, then $(x, x_1, \ldots, x_k, y_l, \ldots, y_1)$ is a path between x and y in $G' \setminus \{w\}$, which is impossible. Thus u is different from x_k and y_l but adjacent to these vertices. But then $(x, x_1, \ldots, x_k, u, y_l, \ldots, y_1)$ is a path from x to y in $G' \setminus \{w\}$, leading again to a contradiction. This shows that w cannot be an articulation point of G'. Since G' is not two-connected, we conclude that u is the only articulation point of G'. By the induction hypothesis, any block B of G' is dominated by some vertex b. Suppose that u does not dominate G', for instance, u is not adjacent to some vertex t of B. Since u is the unique articulation point of G' but is not an articulation point of G, v_1 necessarily has a neighbor $w \neq u$ in B. Hence, there is a path (v_1, w, b, t) of length 3 in $G \setminus \{u\}$ and thus t is a neighbor of u, because $t \in N_3(v_1, G \setminus \{u\}) \subseteq N_1(u)$. Thus u dominates $G' = G \setminus \{v_1\}$, and, since $v_1 \in N_1(u)$, u dominates G as well. This concludes the analysis of Case 2 and the proof of the theorem. □

4. Cop-win graphs for game with witness: class $\bigcap_{k\geq 1} \mathcal{CWW}(k)$

In this and next sections, we investigate the structure of k-winnable graphs. In analogy with big brother graphs, we characterize here the graphs G that are k-winnable for all $k \geq 1$, i.e., the graphs from the intersection $\bigcap_{k\geq 1} \mathcal{CWW}(k)$.

4.1. **Game with witness: preliminaries.** In the k-witness version of the game, the cop first selects his initial position and then the robber selects his initial position which is visible to the cop. As in the classical cop and robber game, the players move alternatively along an edge or pass. However, the robber is visible to the cop only every k moves. After having seen the robber, the cop decides a sequence of his next k moves (the first move of such a sequence is called a *visible* move). The cop captures the robber if they both occupy the same vertex at the same step (even if the robber is invisible). In particular, the cop can capture the visible robber if after the robber shows up, they occupy two adjacent vertices of the graph. Since we are looking for winning strategies for the cop, we may assume that the robber knows the cop's strategy, i.e., after each visible move, the robber knows the next $k-1$ moves of the cop. In the k-witness version of the game, a *strategy* for the cop is a function σ which takes as an input the i first visible positions of the robber and the ik first moves of the cop and outputs the next k moves of the cop. A winning strategy is defined as before and in any k-winnable graph, the cop has a positional winning strategy. We will call a *phase* of the game the movements of the two players comprised between two consecutive visible moves. We will call the behavior of the cop during several consecutive moves of the same phase $\{a, b\}$-*oscillating* if his moves alternate between the adjacent vertices a and b. In a k-winnable graph G, given a winning cop's strategy σ, any trajectory S_r of the robber ends up in a vertex r_p at which the robber is captured. We will say that the trajectory $S_r = (r_1, \ldots, r_p)$ is *maximal* if $(r_1, \ldots, r_{p-1})$ cannot be extended to a longer trajectory for which the robber is not captured by the cop.

Notice that the last vertex r_p in a maximal trajectory S_r corresponds to an invisible move if and only if it is a leaf of G. Indeed, otherwise let r_{p-1} be the previous position of the robber. If $r_{p-1} \neq r_p$, the robber could have stayed in r_{p-1} to avoid being captured. Thus $r_{p-1} = r_p$ and if r_p has at least two neighbors, the robber can safely move to one of the neighbors of r_p not occupied by the cop, and survive for an extra unit of time. We continue with two simple observations, the first shows that during a phase an invisible robber can always safely move around a cycle, while the second shows that a robber visiting one of the vertices a or b during one phase is always captured by an $\{a, b\}$-oscillating cop.

Lemma 1. *Suppose that at his move, the robber $\mathcal{R}$ occupies a vertex v of a cycle C of a graph G and is not visible after this move. Then $\mathcal{R}$ has a move (either staying at v or going to a neighbor of v) such that the cop does not capture the robber during his next move.*

Proof. Let u be a neighbor of v in C which is not occupied by the cop. Since the robber will not be visible after his next move, the strategy of the cop is defined *a priori*. Let z be the next vertex to be occupied by the cop. Then the robber can stay at v if $v \neq z$ or can move to u if $u \neq z$. □

Lemma 2. *If during one phase, the cop is performing $\{a, b\}$-oscillating moves and the robber moves to one of the vertices a or b, then the robber is captured either immediately or at the next move of the cop.*

Proof. Suppose that $\mathcal{R}$ moves to the vertex a. If $\mathcal{C}$ is located at a, then the robber is captured immediately. If $\mathcal{C}$ is located at b and this is not the last vertex of the phase, then $\mathcal{C}$ will move to a and will capture there the robber. Finally, if a and b are the positions of $\mathcal{R}$ and $\mathcal{C}$ at the end of the phase, then the robber will be visible at a and with the next visible move of $\mathcal{C}$ from b to a, the robber will be caught at a. □

4.2. **On the inclusion of $\mathcal{CWW}(k+1)$ in $\mathcal{CWW}(k)$.** Clarke [18] noticed that for any $k \geq 2$, the inclusion $\mathcal{CWFR}(k) \subseteq \mathcal{CWW}(k)$ holds. Contrary to the classes considered in the previous section which collapses for $k \geq 3$, we present now, for each k, an example of a graph in $\mathcal{CWW}(k) \setminus \mathcal{CWW}(k+1)$.

Proposition 8. *For any $k \geq 2$, $\mathcal{CWFR}(k)$ is a proper subclass of $\mathcal{CWW}(k)$. For any $k \geq 1$, there exists a graph contained in $\mathcal{CWW}(k) \setminus \mathcal{CWW}(k+1)$.*

Proof. To see the inclusion $\mathcal{CWFR}(k) \subseteq \mathcal{CWW}(k)$ (which was also mentioned in [18]), it suffices to note that we can interpret the moves at speed k of the robber as if the cop moves only when the robber is visible (i.e., each kth move). Now, let S_3 be the 3-sun, the graph on 6 vertices obtained by gluing a triangle to each of the three edges of another triangle (see Fig. 4(a)). Since no vertex of S_3 has a maximum neighbor, the 3-sun is not dually chordal, thus $S_3 \notin \mathcal{CWFR}(2)$ by Theorem 2. Then clearly, S_3 is not a big brother graph either. On the other hand, $S_3 \in \mathcal{CWW}(k)$ for any $k \geq 2$. Indeed, initially the cop is placed at a vertex u of degree 4. Then, the robber shows himself at the unique vertex v which is not adjacent to

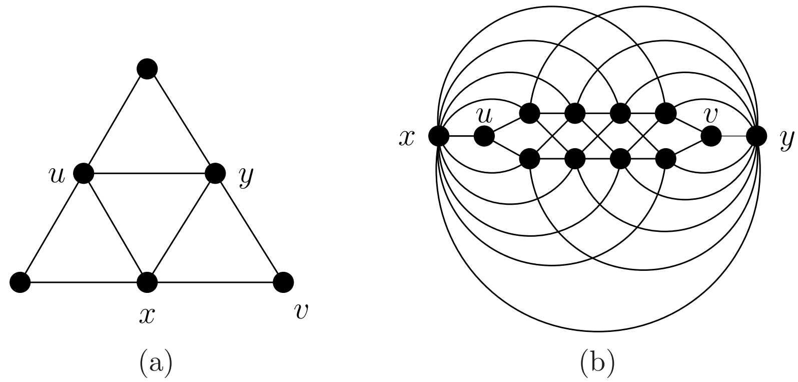


FIGURE 4. Two graphs in (a) $\mathcal{CWW}(k) \setminus \mathcal{CWFR}(k)$, $k \geq 2$ and (b) $\mathcal{CWW}(4) \setminus \mathcal{CWW}(5)$.

u. Let x and y be the two neighbors of v in S_3. The strategy of the cop consists in oscillating between x and y until the robber becomes visible again. Suppose without loss of generality that the cop's sequence of moves is $x, y, x, y, \ldots, y$. Then from Lemma 2 we infer that $\mathcal{R}$ is jammed at vertex v. At the end, when the robber shows his position again, then either he is at v or he desperately moves to x. In both cases, he is caught by $\mathcal{C}$ at the next move. This shows that $\mathcal{CWFR}(k)$ is a proper subclass of $\mathcal{CWW}(k)$

Now we will establish the second assertion. Let $k \geq 1$ and G_k be the graph defined as follows. The vertex set of G_k is $\{x, y, u, v, u_1, \ldots, u_k, v_1, \ldots, v_k\}$. The vertex x is adjacent to any vertex except v, while y is adjacent to any vertex except u. For any $i < k$, the couples $\{u_i, u_{i+1}\}, \{u_i, v_{i+1}\}, \{v_i, v_{i+1}\}, \{v_i, u_{i+1}\}$ are edges of G_k. Finally, u is adjacent to x, u_1, and v_1, while v is adjacent to y, u_k, and v_k (G_4 is depicted in Fig. 4(b)). To prove that $G_k \in \mathcal{CWW}(k)$, consider the following strategy for one cop. Initially, the cop occupies x. To avoid being caught immediatly, the robber must show up at v. The cop occupies alternatively x and y in such a way that after k moves he is at y (if k is odd, then the cop passes his first move). Therefore, after k steps, the robber shows up at a vertex of $N_k(v, G \setminus \{x, y\}) \cup \{x\} \subseteq N_1(y)$, and at the next move the cop caught him. On the other hand, we assert that in G_k a robber with witness $k+1$ can evade against any strategy of the cop. Indeed, assume without loss of generality (in view of symmetry) that the initial position of the cop belongs to the set $L = \{x, u, u_1, \ldots, u_{\lceil k/2 \rceil}, v_1, \ldots, v_{\lfloor k/2 \rfloor}\}$. Then the robber chooses v (or v_1 if $k = 1$ and the cop is occupying u_k) as his initial position. Let z be the vertex occupied by the cop after $k+1$ steps. If $z \in L$, then by Lemma 1 the robber can move in the triangle $\{v, v_k, y\}$ in order to avoid the cop during the $k+1$ steps and to finish at a vertex of the triangle that is not adjacent to z. If $z \notin L$, then the robber uses the $k+1$ steps to reach u (or u_1 if $k = 1$ and $z = v_1$). At any step, there is some $i \leq k$, such that the two vertices u_i and v_i allow the robber to decrease his distance to u (or to u_1) by one; the robber chooses one of these vertices that is not occupied and will not be occupied by the cop after his move. $\square$

Open question 2: Is it true that $\mathcal{CWW}(k+1) \subset \mathcal{CWW}(k)$?

4.3. $\bigcap_{k\geq 1}\mathcal{CWW}(k)$ **and big two-brother graphs.** In analogy to the big brother graphs, we say that a graph G is called a *big two-brother graph*, if G can be represented as an ordered union of subgraphs $G_1,\dots,G_r$ in the form of a tree T rooted at G_1 such that (1) G_1 has a dominating vertex and (2) any $G_i, i>1$, contains one or two adjacent vertices disconnecting G_i from its father and one of these two vertices dominates G_i. Note that if G_i and its father intersect in an articulation point x, then x is not necessarily the vertex which dominates G_i. Equivalently, G is a big two-brother graph if G can be represented as a union of its subgraphs $G_1,\dots,G_r$ labeled in such a way that G_1 has a dominating vertex, and for any $i>1$, either the subgraph G_i intersects $\cup_{j<i}G_j$ in two adjacent vertices x_i,y_i belonging to a common subgraph $G_j, j<i$, so that y_i dominates G_i, or G_i has a dominating vertex y_i and intersects $\cup_{j<i}G_j$ in a single vertex x_i (that may coincide with y_i); we will call such a decomposition $G_1,\dots,G_r$ a *btb-decomposition* of G. The vertices y_i and x_i are the big and the small brothers of G_i. Let $\mathcal{CWW}$ be the class of all big two-brother graphs. See Fig. 3(b) for an example of a big two-brother graph. As for big brother graphs, one can associate a rooted tree T with the decomposition $G_1,\dots,G_r$ of a big two-brother graph G. Obviously any big brother graph G is also a big two-brother graph because the required union of subgraphs is provided by the block decomposition of G and $x_i=y_i$ is the articulation point of the block $G_i=B_i$ relaying it with its father. The 2-trees and, more generally, the chordal graphs in which all minimal separators are vertices or edges are examples of big two-brother graphs which are not big brother graphs.

Theorem 4. *A graph $G=(V,E)$ is k-winnable for all $k\geq 1$ if and only if G is a big two-brother graph, i.e., $\mathcal{CWW}=\bigcap_{k\geq 1}\mathcal{CWW}(k)$.*

Proof. First we show that any big two-brother graph G is k-winnable for any $k\geq 1$. Let $G_1,\dots,G_r$ be a btb-decomposition of G. We consider the following strategy for the cop. The cop starts the game in the big brother of the root graph G_1 and, more generally, at the beginning of each phase, we have the following property: the cop is located in the big brother y_i of some subgraph G_i such that the robber is located in a subgraph G_k that is a descendent of G_i in the decomposition tree T of G. If $G_i=G_k$, then the cop will capture the robber at the first move of the phase. Otherwise, let G_j be the son of G_i on the unique path of T between G_i and G_k. If G_i and G_j intersect in an articulation point x_j, then the cop moves from y_i to x_j, stays there during $k-2$ steps, and then, at the last step of the phase, if x_j is not the big brother y_j of G_j, he moves to y_j. If G_i and G_j intersect in an edge x_jy_j where y_j is the big brother of G_j, then the cop moves from y_i to one of the vertices x_j,y_j and then oscillate between x_j and y_j in such a way that when $\mathcal{R}$ becomes visible again $\mathcal{C}$ occupies the vertex y_j (the decision to move first to x_j or to y_j depends only on the parity of k).

During this phase, the robber cannot leave the subgraph induced by the descendants of G_j, otherwise he has to go from G_j to G_i. In the first case, the cop stays during the whole phase in the unique vertex x_j which cannot be traversed by the robber. In the second case, the cop oscillates between x_j and y_j; therefore, by Lemma 2 the robber cannot traverse $\{x_j,y_j\}$. Therefore, after this phase, the invariant is preserved and the distance in T between the root

and the subgraph G_j hosting the cop has strictly increased. Thus after at most diameter of T phases, $\mathcal{R}$ and $\mathcal{C}$ will be located in the same subgraph G_k, and the cop captures the robber.

Conversely, let $G \in \mathcal{CWW}(k)$ for any $k \geq 1$. If G has a vertex z of degree 1, then $G' = G \setminus \{z\}$ is a retract of G, thus $G' \in \mathcal{CWW}(k)$ for any $k \geq 1$. Hence G' has a btb-decomposition $G_1, \ldots, G_{r-1}$ by induction hypothesis. If w is the unique neighbor of z, then setting G_r to be the edge zw and $y_r = x_r := w$, we will conclude that G is a big two-brother graph as well. So, we can suppose that G does not contain vertices of degree 1.

Since $G \in \mathcal{CWW}(n^2)$, applying Proposition 9 below for $k = n$, where n is the number of vertices of G, we deduce that G contains a vertex v and two adjacent neighbors x, y of v such that $N_n(v, G \setminus \{x, y\}) \subseteq N_1(y)$. This means that the connected component C of $G \setminus \{x, y\}$ containing the vertex v is dominated by y. The graph $G' := G(V \setminus C)$ is a retract of G, thus by Theorem 3 of [18] $G' \in \mathcal{CWW}(k)$ for any $k \geq 1$. By induction assumption, either G' is empty or G' has a btb-decomposition $G_1, \ldots, G_{r-1}$. If G' is empty, then, since y dominates C, we conclude that G has a btb-decomposition consisting of a single subgraph. Otherwise, setting $G_r := G(C \cup \{x, y\})$, $y_r := y$ and $x_r := x$, one can easily see that $G_1, \ldots, G_{r-1}, G_r$ is a btb-decomposition of G. $\square$

Proposition 9. *Let $G \in \mathcal{CWW}(k^2)$ for $k \geq 1$. If the minimum degree of a vertex of G is at least 2, then G contains a vertex v and an edge xy such that $N_k(v, G \setminus \{x, y\}) \subseteq N_1(y)$.*

Proof. If G contains a dominating vertex y, then the result follows by taking as x any vertex of G different from y. Assume thus that G does not have any dominating vertex. Consider a parsimonious winning strategy of the cop and suppose that the robber uses a strategy to avoid being captured as long as possible. Since G does not contain leaves, the robber is caught immediately after having been visible, i.e., at step pk^2+1. Since G does not have dominating vertices, the robber is visible at least twice, i.e. $p \geq 1$. Let y be the vertex occupied by the cop when the robber becomes visible for the last time before his capture. Let v be the next-to-last visible vertex occupied by the robber, i.e., his position at step $(p-1)k^2+1$, and let c_0 be the vertex occupied by the cop at that moment. Finally, let $S_c^p = (c_0, c_1, \ldots, c_{k^2} = y)$ be the trajectory of the cop between the steps $(p-1)k^2+1$ and pk^2+1 (repetitions are allowed). Note that $v \notin N_1(c_0)$, otherwise the robber would have been caught immediately at step $(p-1)k^2+1$. We distinguish two cases depending on whether or not the cop occupies y at least once every two consecutive steps.

Case 1: There exists an index $(p-1)k^2+1 \leq i < pk^2-1$ such that $y \notin \{c_i, c_{i+1}\}$.

Let i be the largest index satisfying the condition of Case 1 and set $x := c_{i+1}$. We will use the following assertion.

Claim 1. *If G contains a cycle C and a vertex $w \in C$ such that $d(v, w) < d(c_1, w) - 1$, then $G \setminus \{x, y\}$ has a connected component that is dominated by y.*

Proof. Let w be a closest to v vertex satisfying the condition of the claim. If the assertion of the claim is not satisfied, we will exhibit a strategy allowing the robber to escape the cop during more steps, contradicting the choice of the strategy of the robber. Suppose that at

the beginning of the pth phase the robber move from v to w along a shortest (v, w)-path. Since $d(v, w) < d(c_1, w)$, the robber cannot be intercepted by the cop during these moves. Suppose that the robber reaches the vertex w before the ith step when the cop arrives at c_i. Then by Lemma 1 the robber can safely move on C until the cop reaches the vertex c_i.

Let z be the position of $\mathcal{R}$ when $\mathcal{C}$ reaches c_i. Then $z \in N_1(y)$, otherwise the robber could stay at z without being caught because starting with this step the cop moves only on vertices of $N_1(y)$. Suppose that there exists a vertex t at distance 2 from y in $G \setminus \{x\}$. Let $r \neq x$ be a common neighbor of t and y. The following sequence of moves is valid for the robber: when the cop is in c_i, the robber goes from z to y (or stays in y, if $z = y$); once the cop has moved to $x = c_{i+1}$, the robber goes from y to r; finally, once the cop has moved to y, the robber goes from r to t. After this step, by definition of c_i, the cop only stays in $N_1(y)$ and finishes in y. Hence, the robber can remain in t and will not be captured the next time he shows up, a contradiction. This concludes the proof of the claim. □

If the vertex v belongs to a cycle C, then setting $w := v$ and applying Claim 1 we conclude that y dominates the connected component of $G \setminus \{x, y\}$ containing v, establishing thus the assertion of Proposition 9. So, suppose that v is an articulation point of G not contained in a cycle. Since the minimum degree of G is at least 2, $G \setminus \{v\}$ has a connected component D that does not contain c_0 (nor c_1). Necessarily D contains a cycle C, otherwise we will find in D a vertex of degree 1 in G. Since any path from c_1 to a vertex w of C passes via v and c_1 is not adjacent to v, we obtain $d(v, w) < d(c_1, w) - 1$. The result then follows from the claim. This concludes the analysis of Case 1.

Case 2: For any $(p-1)k^2 \leq i \leq pk^2$ we have $y \in \{c_i, c_{i+1}\}$, i.e., $\mathcal{C}$ occupies y at least once every 2 steps.

First, assume that there exists a vertex x (possibly $x = y$) and $(p-1)k^2 \leq i \leq pk^2 - k$ such that $c_i, \ldots, c_{i+k} \in \{y, x\}$, i.e., that there are at least k consecutive steps when the cop remains at x or y. Then, we claim that $N_k(v, G \setminus \{x, y\}) \subseteq N_1(y)$. Indeed, pick $z \in N_k(v, G \setminus \{x, y\})$ and let $P = (v = p_1, \ldots, p_k = z)$ be a shortest path in $G \setminus \{x, y\}$ between v and z. Until the ith step of the phase, the robber may progress "slowly" along P: either by staying at his current position, or moving to the next vertex of P toward z, depending on the moves of the cop. The cop starts oscillating between x and y at step i. Then during the next k steps, the robber can follow P until he reaches z (since the length of P is at most k). Therefore, if z is not a neighbor of y, then the robber can remain at z until step k^2p without being captured. Since by our assumption the robber is caught at step k^2p, necessarily $z \in N_1(y)$. Hence $N_k(v, G \setminus \{x, y\}) \in N_1(y)$ and the assertion of Proposition 9 holds.

Therefore, we may assume that between the steps $(p-1)k^2$ and pk^2, for all k consecutive steps, the cop occupies at least three distinct vertices (one of which is y). We assert in this case that $N_k(v, G \setminus \{y\}) \subseteq N_1(y)$. Pick $z \in N_k(v, G \setminus \{y\})$ and let P be a shortest path between v and z in $G \setminus \{y\}$. Then for any vertex w of P, among any sequence of k moves of the cop we can find three consecutive moves during which the cop does not occupy w. Therefore, for any sequence of k consecutive steps the robber can reduce by one his distance to z by

moving on P towards z without being captured. Hence, he will reach z before step pk^2. If z is not adjacent to y, then staying at z the robber will not be captured, a contradiction. This concludes the proofs of Proposition 9 and Theorem 4. □

5. Cop-win graphs for game with witness: classes $\mathcal{CWW}(k)$

In this section we investigate the dismantling orders related to k-winnable graphs. We provide a dismantling order which must be satisfied by all graphs of the class $\mathcal{CWW}(2)$. We show that this order is not sufficient but some its reinforcement is. Then we continue with similar results about k-winnable graphs for odd values of $k \geq 3$.

5.1. **Class $\mathcal{CWW}(2)$.** We continue with the definition of a dismantling ordering which seems to be intimately related with the witness variant of the cop and robber game. Again, we will consider a slightly more general version of the game: given a subset of vertices X of a graph $G = (V, E)$, the *X-restricted k-witness game* of cop and robber, is a variant in which $\mathcal{R}$ can pass through any vertex of G, $\mathcal{C}$ can move only inside X, and all visible positions of the robber are at vertices of X. Then X is called *k-winnable* if for any starting positions of $\mathcal{C}$ and $\mathcal{R}$, the cop wins in the X-restricted variant of the k-witness version of the game. We will say that a subset of vertices X of a graph $G = (V, E)$ is *k-bidismantlable* if the vertices of X can be ordered $v_1, \ldots, v_m$ in such a way that for each vertex $v_i, 1 \leq i < m$, there exist two adjacent or coinciding vertices x, y with $y = v_j, x = v_\ell$ and $j, \ell > i$ such that $N_k(v_i, G \setminus \{x, y\}) \cap X_i \subseteq N_1(y)$, where $X_i := \{v_i, v_{i+1}, \ldots, v_m\}$ and $X_m = \{v_m\}$. We say that a graph $G = (V, E)$ is *k-bidismantlable* if its vertex-set V is k-bidismantlable. In case $k = 2$, the inclusion $N_2(v_i, G \setminus \{x, y\}) \cap X_i \subseteq N_1(y)$, can be equivalently written as $N_2(v_i, G \setminus \{x\}) \cap X_i \subseteq N_1(y)$. Any $(k, 1)$-dismantlable graph is k-bidismantlable but the converse is not true: for any $k \geq 2$, the 3-sun S_3 presented in Fig. 4 is k-bidismantlable but not $(k, 1)$-dismantlable. In some proofs, we will denote by $x(v)$ and $y(v)$ the vertices eliminating a vertex v in a k-bidismantling order.

Proposition 10. *Any graph $G = (V, E)$ of $\mathcal{CWW}(2)$ is 2-bidismantlable.*

Proof. Suppose that a subset $X \subseteq V$ is 2-winnable and assume that there exists an order $u_1, \ldots u_\ell$ on the vertices of $V \setminus X$ such that for each $1 \leq i \leq \ell$, there exist the vertices $x(u_i), y(u_i) \in X_{i+1}$ such that $N_2(u_i, G \setminus \{x(u_i), y(u_i)\}) \cap X_i \subseteq N_1(y(u_i))$ holds, where $X_i = \{u_i, \ldots, u_\ell\} \cup X$. We show by induction on the size of X that the set X is 2-bidismantlable. Assume $|X| \geq 2$, otherwise, X is trivially 2-bidismantlable. We first show that we can select a vertex $v_1 \in X$, a vertex $y \in N(v_1) \cap X$, $y \neq v_1$, and a vertex $x \in N_1(y) \cap N(v_1) \cap X$ such that $N_2(v_1, G \setminus \{x, y\}) \cap X \subseteq N_1(y)$. If there exists a vertex $y \in X$ such that $X \subseteq N_1(y)$, then taking $x := y$ and any vertex of $X \setminus \{y\}$ as v_1, we are done. So, further we assume that X does not contain dominating vertices.

Consider a parsimonious winning strategy of the cop and a maximal trajectory of the robber. First suppose that the capture happened when $\mathcal{R}$ is invisible. Let v_1 be the last position where the robber is visible. Let a be the position of the cop when the robber shows

up in v_1. We know that $v_1 \notin N(a)$, otherwise the cop would have captured the robber before. Let y be the vertex where $\mathcal{C}$ moves when he sees $\mathcal{R}$ in v_1. Since the robber is captured when he is invisible, it implies he is captured in v_1. Moreover, since the robber follows a maximal trajectory, it implies that $N_2(v_1, G \setminus \{y\}) \cap X = \{v_1\}$, otherwise the robber could live longer. Consequently, by setting $x := y$, we have $N_2(v_1, G \setminus \{x, y\}) \cap X \subseteq N_1(y)$.

Now suppose that $\mathcal{C}$ captures $\mathcal{R}$ at the next visible move. This means that when $\mathcal{C}$ sees $\mathcal{R}$, the cop is located in some vertex $y \in X$ and the robber is located in some vertex $w \in X$ and $w \in N_1(y)$ holds. Then the cop moves from y to w and captures $\mathcal{R}$ there. Denote by v_1 the vertex of X where $\mathcal{R}$ is visible for the next-to-last time. Suppose that after having seen the robber in v_1, the cop moves first to a vertex of X which we denote by x and then to vertex y. Note that $x \neq v_1$ (otherwise the robber would have been caught when he shows up in v_1) and that y may coincide with x or with v_1. When the cop moves to x, the robber first moves to some vertex $u \in N_1(v_1) \setminus \{x\}$ and then, when $\mathcal{C}$ moves to y, $\mathcal{R}$ moves to a vertex $w \in N_1(u) \cap X \subseteq (N_2(v_1, G \setminus \{x\}) \cup \{x\})$. By the definition of the vertices y and w, in y the cop sees (for the last time) the robber which is located at w and with the next move captures him. Since $\mathcal{R}$ follows a maximal sequence of moves before his capture, any vertex of $N_2(v_1, G \setminus \{x\}) \cap X$ must be adjacent to y, otherwise, if there exists $z \in N_2(v_1, G \setminus \{x\}) \cap X$ not adjacent to y, instead of moving to w, in two moves the robber can safely reach z and survive for a longer time. Thus $N_2(v_1, G \setminus \{x\}) \cap X \subseteq N_1(y)$ holds.

If $v_1 \neq y$, then we are done. If $v_1 = y$, then $N_2(y, G \setminus \{x\}) \cap X \subseteq N_1(y)$. If $N_1(y) \cap X \subseteq N_1(x)$, then $N_2(v_1, G \setminus \{x\}) \cap X \subseteq N_1(y) \cap X \subseteq N_1(x)$ and thus by setting $y(v_1) := x(v_1) := x$, we have $N_2(v_1, G \setminus \{x(v_1), y(v_1)\}) \cap X \subseteq N_1(y(v_1))$ and again we are done. Suppose now that there exists a vertex $v \in N_1(y) \cap X$ which does not belong to $N_1(x)$. We assert that $N_2(v, G \setminus \{x, y\}) \cap X \subseteq N_1(y)$. Since $N_1(v, G \setminus \{x, y\}) \cap X \subseteq N_2(y, G \setminus \{x\}) \cap X \subseteq N_1(y)$, any neighbor u of v in X is a neighbor of y. Consider a vertex $u \in N_2(v, G \setminus \{x, y\}) \cap X$ and suppose there exists a vertex $r \in N_1(v) \cap N_1(u) \cap X \setminus \{x, y\}$. Then $r \in N_1(y)$ and thus $u \in N_2(y, G \setminus \{x\}) \cap X \subseteq N_1(y)$. Suppose now that there does not exist any vertex $r \in N_1(v) \cap N_1(u) \setminus \{x, y\}$ that belongs to X. Among all vertices in $N_1(v) \cap N_1(u) \setminus \{x, y\}$, let r be the last vertex occurring in the ordering $u_1, \ldots, u_\ell$. Then, since $u, v \in N_1(r) \cap X$, $u, v \in N_1(y(r))$ and consequently, $y(r) \neq x$, since $v \notin N_1(x)$. By our choice of r, we know that $y(r) \in X$ and thus there exists a vertex in $N(v) \cap N(u) \cap X \setminus \{x, y\}$, a contradiction. Therefore, by setting $x(v) := y(v) := y$, we have $N_2(v, G \setminus \{x(v), y(v)\}) \cap X \subseteq N_1(y(v))$. In the rest of the proof, we denote by v_1 the vertex satisfying this condition, it can be either v_1 or v.

Consider the set $X' := X \setminus \{v_1\}$. Note that $V \setminus X' = V \setminus X \cup \{v_1\}$, and there exists an order $u_1, \ldots u_\ell, u_{\ell+1} := v_1$ on the vertices of $V \setminus X'$ such that for each $1 \leq i \leq \ell + 1$, there exist $x(u_i), y(u_i) \in X_{i+1}$ such that $N_2(u_i, G \setminus \{x(u_i), y(u_i)\}) \cap X_i \subseteq N_1(y(u_i))$. We show that the set X' is 2-winnable as well. Consider a positional parsimonious winning strategy σ of the cop in X. For any positions c of the cop and r of the robber in X', we note $\sigma(c, r) = (c_1, c_2)$. As in the proof of Theorem 1, we construct a strategy that uses one bit of memory m: it is a function that associates to each (c, r, m) a couple $((c'_1, c'_2), m)$. As in the proof of Theorem 1,

the intuitive idea is that the cop plays using σ, except when he is in y and his memory contains 1; in that case, he plays using σ as if he was in v_1.

If $m = 0$ or $c \neq y$, let $(c_1, c_2) = \sigma(c, r)$. If $c_1 = v_1$, then $c'_1 = y$ and $c'_1 = c_1$ otherwise. If $c_2 = v_1$, then $\sigma'(c, r, m) = ((c'_1, y), 1)$ and $\sigma'(c, r, m) = ((c'_1, c_2), 0)$ otherwise. If $m = 1$ and $c = y$, let $(c_1, c_2) = \sigma(v_1, r)$. If $c_1 = v_1$, then $c'_1 = y$ and $c'_1 = c_1$ otherwise. If $c_2 = v_1$, then $\sigma'(y, r, 1) = ((c'_1, y), 1)$ and $\sigma'(y, r, 1) = ((c'_1, c_2), 0)$ otherwise. Since $N_1(v_1) \cap X \subseteq N_1(y)$, one can easily check that σ' is a valid strategy for the X'-restricted game.

By way of contradiction, suppose now that there exists an infinite X'-valid sequence S'_r of moves of the robber in the X'-restricted game allowing him to escape forever against a cop using the strategy σ'. First note that the sequence of moves S_c of the cop playing σ against S'_r differs from the sequence of moves S'_c of the cop playing σ' against S'_r only in the positions where the cop is in v_1 in S_c.

We show that there exists an infinite sequence S_r in the X-restricted game enabling the robber to escape forever against a cop using the strategy σ. The visible positions of $\mathcal{R}$ in S_r will coincide with the visible positions of $\mathcal{R}$ in S'_r (thus the cop's strategies σ and σ' behave in the same way against both sequences). It is sufficient to show that if during a phase of S'_r, the robber goes from $r'_0 \in X'$ to $r'_2 \in X'$ via $r'_1 \in V(G)$, then in the X-restricted game where the cop plays with strategy σ (going first to c_1 and then to c_2), there exists r_1 such that $\mathcal{R}$ can go from r'_0 to r'_2 via r_1 without being captured in r_1.

If $r'_1 \neq v_1$ or if $v_1 \notin \{c_1, c_2\}$, then one can choose $r_1 = r'_1$ (since $r'_0, r'_2 \in X'$, they are different from v_1). Thus, we may assume that $r'_1 = v_1$ and that $c_1 = v_1$ or $c_2 = v_1$. If $c_2 \in \{v_1, y\}$, then $c'_2 = y$. Since $r'_1 = v_1$, $r'_2 \in N_1(v_1) \cap X \subseteq N_1(y)$ and thus the robber is captured when he shows up in r'_2, i.e., S'_r does not enable the robber to escape forever. Consequently, $c_2 \notin \{v_1, y\}$ and $c_1 = v_1$. In this case, $(r'_0, r_1 := y, r'_2)$ is a X-valid sequence since $r'_0, r'_2 \in N_1(v_1) \cap X \subseteq N_1(y)$ and moreover $y \notin \{c_1, c_2\}$ (since $c_1 = v_1$ and $y \neq c_2$). It implies that there exists an infinite X-valid sequence S_r enabling the robber to escape forever, a contradiction.

Starting from a positional strategy for the X-restricted game, we have constructed a winning strategy using memory for the X'-restricted game. As mentioned in the introduction, it implies that there exists a positional winning strategy for the X'-restricted game. Consequently, the set $X' := X \setminus \{v_1\}$ is 2-winnable as well. By induction assumption, X' admits a 2-bidismantling order $v_2, \ldots, v_m$. Then clearly $v_1, v_2, \ldots, v_m$ is a 2-bidismantling of X. If G is 2-winnable, then its set of vertices is 2-winnable and therefore 2-bidismantlable, showing that G is 2-bidismantlable. $\square$

We continue with two examples. The first one shows that we cannot replace in the definition of 2-bidismantlability the condition $N_2(v_i, G \setminus \{x\}) \cap X_i \subseteq N_1(y)$ by a weaker condition $N_2(v_i, G_i \setminus \{x\}) \subseteq N_1(y)$ (i.e., instead of all vertices of X_i reachable from v_i by paths of length 2 avoiding x of the whole graph G to consider only the vertices reachable by such paths of the subgraph G_i). The second example shows that unfortunately 2-bidismantlability is not a sufficient condition.

Proposition 11. *Let G be the graph from Fig. 5. Then G admit a dismantling order satisfying the condition $N_2(v_i, G_i \setminus \{x\}) \subseteq N_1(y)$, however G is not 2-bidismantlable nor 2-winnable.*

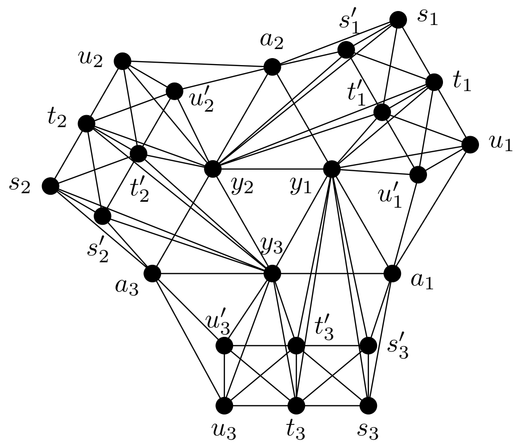


FIGURE 5. A weakly 2-bidismantlable graph that is not 2-bidismantlable

Proof. Consider the following order on the vertices of G: a_1, a_2, a_3, u_1, u_1', u_2', u_3, u_3', s_1, s_1', s_2, s_2', s_3, s_3', t_1, t_1', t_2, t_2', t_3', t_3', y_1, y_2, y_3. For each vertex $v \in V(G) \setminus \{y_3\}$, we give below two adjacent vertices $x(v), y(v)$ that are eliminated later than v and such that $N_2(v, G_i \setminus \{x(v), y(v)\}) \subseteq N_1(y(v))$ (the vertex $x(v)$ is not defined if $N_2(v, G_i \setminus \{y(v)\}) \subseteq N_1(y(v)))$.

v	a_1	a_2	a_3	u_1	u_1'	u_2	u_2'	u_3	u_3'	s_1	s_1'	s_2
$y(v)$	y_1	y_2	y_3	t_1	t_1	t_2	t_2	t_3	t_3	y_2	y_2	y_3
$x(v)$	y_3	y_1	y_2	y_1	y_1	y_2	y_2	y_3	y_3	–	–	–
v	s_2'	s_3	s_3'	t_1	t_1'	t_2	t_2'	t_3	t_3'	y_1	y_2	y_3
$y(v)$	y_3	y_1	y_1	y_2	y_2	y_3	y_3	y_1	y_1	y_2	y_3	–
$x(v)$	–	–	–	y_1	y_1	–	–	–	–	–	–	–

We prove now that G is not 2-bidismantlable. Note that for a_1 (resp. a_2, a_3), there exist $y(a_1) = y_1$ (resp. $y(a_2) = y_2, y(a_3) = y_3$) and $x(a_1) = y_3$ (resp. $x(a_2) = y_1, x(a_3) = y_2$) such that $N_2(a_1, G \setminus \{y_1, y_2\}) \subseteq N_1(y_1)$ (resp. $N_2(a_2, G \setminus \{y_2, y_3\}) \subseteq N_1(y_2)$, $N_2(a_3, G \setminus \{y_3, y_1\}) \subseteq N_1(y_3)$). Consequently, any 2-bidismantling order of G can start with a_1, a_2, a_3. In fact, one can check that any 2-bidismantling of G must start with a permutation of a_1, a_2, a_3. We will show now that it is impossible to extend a 2-bidismantling order starting with a_1, a_2, a_3. To prove this, it suffices to show that for any $v \in V(G) \setminus \{a_1, a_2, a_3\}$ and for all adjacent vertices $x(v), y(v) \in N_1(v)$, there exists a vertex $z(v) \in N_2(v, G \setminus \{x(v), y(v)\}) \setminus \{a_1, a_2, a_3\}$ such that $z(v) \notin N_1(y(v))$. In view of symmetry of G, it is sufficient to check this property for $v \in \{u_1, t_1, y_1\}$.

If $v = u_1$, then $y(u_1), x(u_1) \in \{a_1, u'_1, t_1, t'_1, y_1\}$. If $y(u_1) \in \{a_1, u'_1, y_1\}$, then either $t_1 \neq x(v_1)$, or $t'_1 \neq x(v_1)$. In both cases, $s_1 \in N_2(v, G \setminus \{x(u_1), y(u_1)\})$ and $s_1 \notin N_1(y(u_1))$. By symmetry, we can suppose that $y(u_1) = t_1$. Since $a_1 \notin N_1(t_1)$, we must have $x(u_1) \neq a_1$ and consequently, $s_3 \in N_2(v, G \setminus \{x(u_1), y(u_1)\})$ and $s_3 \notin N_1(y(u_1))$.

If $v = t_1$, then $y(t_1), x(t_1) \in \{u_1, u'_1, s_1, s'_1, t'_1, y_1, y_2\}$. If $y(t_1) \in \{y_1, u_1, u'_1\}$ (resp. $y(t_1) = \{y_2, s_1, s'_1\}$), then set $z(v) = s_1$ (resp. $z(v) = u_1$); in all cases, $z(v) \in N_2(v, G \setminus \{x(t_1), y(t_1)\})$ and $z(v) \notin N_1(y(t_1))$. If $y(t_1) = t'_1$, then either $x(t_1) \neq y_2$ or $x(t_1) \neq y_1$; in both cases, $y_3 \in N_2(v, G \setminus \{x(t_1), y(t_1)\})$ and $y_3 \notin N_1(y(t_1))$.

If $v = y_1$, since $N_1(y_1) \subseteq N_1(y(y_1))$, the vertex $y(y_1)$ must belong to $N_1(y_1) \cap N_1(y_2) \cap N_1(y_3)$. Consequently, by symmetry, we can assume that $y(y_1) = y_2$. However, since $u_1 \in N_1(y_1) \setminus N_1(y_2)$, we obtain $u_1 \in N_2(y_1, G \setminus \{x(y_1), y(y_1)\})$ and $u_1 \notin N_1(y(y_1))$. This completes the proof that G is not 2-bidismantlable. Since any graph $G \in \mathcal{CWW}(2)$ is 2-bidismantlable, it also implies that $G \notin \mathcal{CWW}(2)$. $\square$

Proposition 12. *Let G be the graph from Fig. 6. Then G is 2-bidismantlable, however $G \notin \mathcal{CWW}(2)$.*

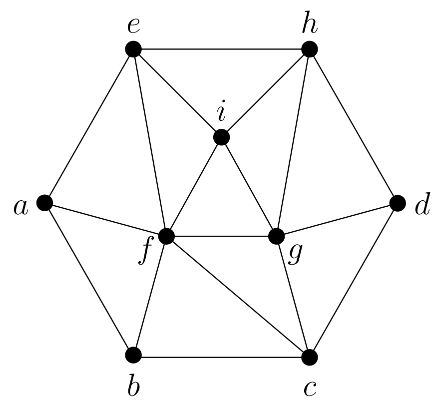


FIGURE 6. A 2-bidismantlable graph $G \notin \mathcal{CWW}(2)$

Proof. The graph presented in Fig. 6 is 2-bidismantlable with the following 2-bidismantling order $a, b, c, d, e, f, g, h, i$, where each vertex v is eliminated by the vertices $x(v)$, $y(v)$ defined as follows:

v	a	b	c	d	e	f	g	h	i
$y(v)$	f	f	g	g	i	i	i	i	$-$
$x(v)$	e	c	f	h	$-$	$-$	$-$	$-$	$-$

However, one can show that for any vertex c there exists a vertex r such that, if at step $i \geq 0$ the cop moves to (or starts in) c (going through any intermediate vertex), then the robber can move to (or starts in) r without being caught. Since for any such couple (c, r)

the vertices c and r are not adjacent, it means that the cop cannot catch the robber in this graph. The definition of the pairs (c, r) is given in the following table:

c	a	b	c	d	e	f	g	h	i
r	d	d	e	e	b	d	e	b	b

Note that if the robber wants to go from d to e, (resp., from e to d), then this means that the cop is in a, b or f (resp., wants to go to a, b or f). Since $h \notin N(a) \cup N(b) \cup N(f)$, h cannot be the intermediate vertex used by the cop. Thus, the robber can always go from d to e (resp. from e to d) via h.

If the robber wants to go from b to d (resp., from d to b), then this implies that the cop is in e, h, i (resp., wants to go to e, h, i). Since $c \notin N(e) \cup N(h) \cup N(i)$, c cannot be the intermediate vertex used by the cop. Thus, the robber can always go from b to d (resp. from d to b) through c.

If the robber wants to go from b to e (resp. from e to b), then this means that the cop neither starts in a nor f (because in this case the robber would have been in d), nor goes to a or f (since in this case, the robber wants to go in d). Moreover, the intermediate vertex used by the cop is different from a or f. In the first case (resp. second case), the robber can go from b to e via a (resp. f). $\square$

We continue with a condition on 2-bidismantling which turns out to be sufficient for 2-winability. We say that a graph G is *strongly 2-bidismantlable* if G admits a 2-bidismantling order such that for any vertex $v_i, i < n$, $y(v_i) = x(v_i)$ or $N_2(v_i, G \setminus \{y(v_i)\}) \cap X_i \subseteq N_2(x(v_i), G \setminus \{y(v_i)\})$ (recall that $x(v)$ and $y(v)$ denote the vertices eliminating a vertex v in a 2-bidismantling order).

Proposition 13. *If a graph G is strongly 2-bidismantlable, then $G \in \mathcal{CWW}(2)$.*

Proof. Suppose that a subset X of vertices of G admits a strong 2-bidismantling order $v_1, \ldots, v_m$. Assume by induction assumption that the set $X' = \{v_2, \ldots, v_n\}$ is 2-winnable and we will establish that the set X itself is 2-winnable. Let $N_2(v_1, G \setminus \{x\}) \cap X \subseteq N_1(y)$. Let σ' be a parsimonious positional winning strategy for $\mathcal{C}$ in X'. We define the strategy σ for $\mathcal{C}$ in X as follows: $\sigma(c, r) = r$ if $r \in N_1(c)$, $\sigma(c, v_1) = (x, y)$ if $c \in N_1(x)$ (in this case, the robber will be caught during the next move because $N_2(v_1, G \setminus \{x\}) \cup X \subseteq N_1(y)$) and $\sigma(c, v_1) = \sigma'(c, x)$ otherwise, and $\sigma(c, v) = \sigma'(c, v)$ in all other cases. We now prove that σ is winning. Let $S_r = (r_1, r_2, \ldots)$ be any X-valid sequence of moves of the robber. We will transform S_r into a X'-valid sequence $S'_r = (r'_1, r'_2, \ldots)$ of moves of the robber and prove that, since $\mathcal{C}$ playing σ' eventually captures $\mathcal{R}$ following S'_r, then $\mathcal{C}$ playing σ captures $\mathcal{R}$ following S_r.

Let $r'_1 := x$ if $r_1 = v_1$ and $r'_1 := r_1$ otherwise. Suppose that $r'_1, \ldots, r'_{2j-1}$ ($j \geq 1$) have been already defined and we wish to define r'_{2j} and r'_{2j+1}. We set $r'_{2j+1} := r_{2j+1}$ if $r_{2j+1} \neq v_1$ and $r'_{2j+1} := x$ otherwise (indeed, when the cop sees the robber in the vertex v_1, then $\mathcal{C}$ will plays against $\mathcal{R}$ as like the latter was in x). We set $r'_{2j} := r_{2j}$ in all cases unless $v_1 \in \{r_{2j-1}, r_{2j+1}\}$

and $r_{2j} \notin N_1(x)$ (in particular $r_{2j} \neq y$). If $r_{2j-1} = v_1$ (resp., if $r_{2j+1} = v_1$) and $r_{2j} \notin N_1(x)$, then there exists a common neighbor u of r_{2j-1} (resp., r_{2j+1}) and x different from y. The choice of r'_{2j} depends of the current position c_{2j} of the cop pursuing $\mathcal{R}$. We set $r'_{2j} := u$ if $c_{2j} \neq u$ and $r'_{2j} := y$ otherwise (this is to avoid to artificially create a move where the robber goes to a vertex occupied by the cop). It can be easily seen that S'_r is a X'-valid sequence of moves of the robber.

Let $S'_c = (c'_1, c'_2, \ldots)$ be the X'-valid sequence of moves of the cop playing σ' against a robber $\mathcal{R}'$ moving according to S'_r, and let $S_c = (c_1, c_2, \ldots)$ be the X-valid sequence of moves of the cop playing σ against the robber $\mathcal{R}$ following S_r. It is easy to check that S'_c and S_c are similar except one or two steps before the capture of the robber. Moreover, since σ' is a winning strategy in X', there is $j > 0$ such that $c'_j = r'_j$.

First suppose that $\mathcal{C}$ captures the robber $\mathcal{R}'$ when he is visible, say $\mathcal{R}'$ is located in r'_{2j+1}. If $r'_{2j+1} = r_{2j+1}$, then we are done. So, suppose that $r'_{2j+1} \neq r_{2j+1}$, i.e., $r_{2j+1} = v_1$ and $r'_{2j+1} = x$. Therefore, when $\mathcal{C}$ sees $\mathcal{R}$ in v_1, the cop is located in a neighbor of x. According to σ, $\mathcal{C}$ will move to x and then to y, while $\mathcal{R}$ can only reach a vertex in $N_2(v_1, G \setminus \{x\}) \cap X$. Since $N_2(v_1, G \setminus \{x\}) \cap X \subseteq N_1(y)$, the cop will capture the visible robber at his next move.

Now suppose that $\mathcal{C}$ captures $\mathcal{R}'$ when the latter is invisible, say $\mathcal{R}'$ is located in r'_{2j}. Again, if $r'_{2j} = r_{2j}$, then we are done. Otherwise, according to the definition of S'_r, we conclude that r_{2j} is a common neighbor of r_{2j-1} and r_{2j+1} different from y with either $v_1 = r_{2j+1}$ or $v_1 = r_{2j-1}$. Suppose that $v_1 = r_{2j+1}$ (the other case is analogous), r'_{2j} is either y or a common neighbor u of r_{2j-1} and x provided by the strong 2-bidismantling order. Since, between r_{2j-1} and $r_{2j+1} = v_1$ the trajectory of $\mathcal{R}'$ avoids the cop if possible, we deduce that $\{c_{2j-1}, c_{2j}\} = \{u, y\}$ or $\{c_{2j}, c_{2j+1}\} = \{u, y\}$. If $\{c_{2j-1}, c_{2j}\} = \{u, y\}$, then, when $\mathcal{C}$ sees $\mathcal{R}$ in r_{2j-1}, the cop is located in a neighbor of r_{2j-1}. By the definition of σ, $\mathcal{C}$ will move to r_{2j-1} and captures $\mathcal{R}$. Otherwise, if $\{c_{2j}, c_{2j+1}\} = \{u, y\}$, then when the cop sees $\mathcal{R}$ in v_1, $\mathcal{C}$ is located in a neighbor of x. By the definition of σ, as before, $\mathcal{C}$ will move to x and then to y, while $\mathcal{R}$ can only reach a vertex in $N_2(v_1, G \setminus \{x\}) \cup X$. Since $N_2(v_1, G \setminus \{x\}) \cap X \subseteq N_1(y)$, the cop will capture the visible robber at his next move. □

We conclude this section by showing that the existence of a strong 2-bidismantling order is not necessary.

Proposition 14. *The graph G from Fig. 7 belongs to $\mathcal{CWW}(2)$, however G does not admit a strong 2-bidismantling order.*

Proof. We first show that the graph G from Fig. 7 is in $\mathcal{CWW}(2)$. The cop starts in u. Hence, if the robber starts in x, x', y_1 or y_2, he is immediately caught. If the robber starts in s_1 (or s_2), then the cop moves to y_1 (resp., to y_2) and since $N_2(s_1) \subseteq N_1(y_1)$ (resp., $N_2(s_2) \subseteq N_1(y_2)$), the robber is caught the next time he shows up. If the robber starts in v_1 (the cases v'_1, v_2, v'_2 are similar), then the cop first moves to x and then to y_1. Then the robber has to show up in a vertex of $\{v_1, s_1, v'_1, x\} \subseteq N_1(y_1)$ and the cop can catch him.

Consider now any 2-bidismantling order of G. Let v be the first vertex in this order which is different from s_1, s_2. We may assume without loss of generality that $v \in \{v_1, x, y_1, u\}$. Let

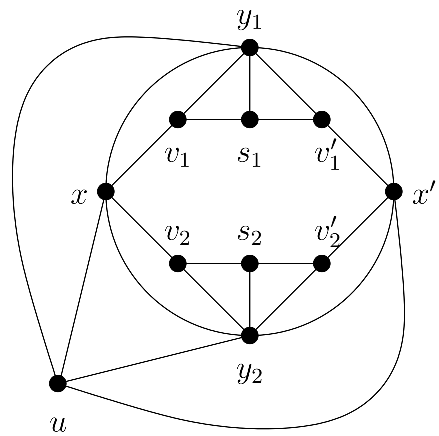


FIGURE 7. A graph $G \in \mathcal{CWW}(2)$ which is not strongly 2-bidismantlable.

$X = V(G)\setminus\{s_1, s_2\}$. Since there does not exist t such that the set $N_1(x)\cap X$ (resp. $N_1(y_1)\cap X$ or $N_1(u)\cap X$) is included in $N_1(t)$, it implies $v = v_1$. We know that $x(v_1), y(v_1) \in N_1(v_1)$ and that $N_1(v_1) \cap X \subseteq N_1(y(v_1))$. Consequently, $y(v_1) \in \{x, y_1\}$. If $y(v_1) = x$, then $x(v_1) \neq s_1$ (since s_1 and x are not adjacent) and thus $v_1' \in N_2(v_1, G \setminus \{x(v_1), y(v_1)\}) \setminus N_1(y(v_1))$, which is impossible. Thus, $y(v_1) = y_1$. If $x(v_1) \neq x$, then $v_2 \in N_2(v_1, G \setminus \{x(v_1), y(v_1)\}) \setminus N_1(y_1)$. Consequently, $y(v_1) = y_1$ and $x(v_1) = x$. However, $v_1' \in N_2(v_1) \cap X$ but $v_1' \notin N_2(x, G \setminus \{y_1\})$ and thus G is not strongly 2-bidismantlable. □

5.2. **Classes $\mathcal{CWW}(k)$ for $k \geq 3$.** In this subsection we show that k-bidismantlable graphs are k-winnable for any odd $k \geq 3$. We also show that for any $k \geq 3$, there exist graphs in $\mathcal{CWW}(k)$ that are not k-bidismantlable, i.e., for $k \geq 3$, k-bidismantlability of a graph is not a necessary condition to be k-winnable.

Theorem 5. *For any odd integer $k \geq 3$, if a graph G is k-bidismantlable, then $G \in \mathcal{CWW}(k)$.*

Proof. Suppose that $X \subseteq V$ is a k-bidismantlable set of vertices of a graph G. We prove that there is a winning strategy for the cop in the X-restricted k-witness game on G. To do so, we proceed as in the papers [27, 30] and use the k-bidismantling order to mark all X-configurations (c, r). A *X-configuration* of X-restricted game is a couple (c, r) that consists of a position of the cop $c \in X$ and a position of the robber $r \in X$, with $r \neq c$. A X-configuration (c, r) is called *terminal* if $r \in N_1(c)$.

To mark the X-configurations, we use the following procedure `Mark(X)`.

(1) Initially, all X-configurations are unmarked.
(2) Any *terminal* X-configuration (c, r) is marked with label 1.
(3) While it is possible, mark an unmarked X-configuration (c, r) with the smallest possible integer $\ell + 1$ such that there exist vertices $y_{(c,r)} \in N_1(c) \cap X$ and

$x_{(c,r)} \in (N_1(y_{(c,r)}) \setminus \{r\}) \cap X$ such that for all $z \in N_k(r, G \setminus \{x_{(c,r)}, y_{(c,r)}\}) \cap X$, the X-configuration $(y_{(c,r)}, z)$ is marked with a label at most ℓ.

Claim 2. *If all X-configurations are marked by* `Mark(`X`)`*, then there is a winning strategy for the cop in the X-restricted k-witness game on G.*

Indeed, pick any initial positions $c \in X$ of the cop and $r \in X$ of the robber. If the configuration (c, r) is terminal, then $r \in N_1(c)$ and the robber is captured at the next move. Otherwise, the cop first moves to $y_{(c,r)}$ and then oscillates between $x_{(c,r)}$ and $y_{(c,r)}$ during $k-1$ steps, i.e., the cop ends in $y_{(c,r)}$ since k is odd. If during one of his invisible moves the robber goes to $x_{(c,r)}$ or $y_{(c,r)}$, then he will be captured immediately. Otherwise, in k moves the robber goes from r to a vertex $z \in N_k(r, G \setminus \{x_{(c,r)}, y_{(c,r)}\}) \cap X$. According to `Mark(`$X$`)`, the label of $(y_{(c,r)}, z)$ is strictly less than that of (c, r). Therefore, by repeating the same process, after a finite number of steps either the cop captures the robber during an invisible move or the cop and the robber arrive at a terminal configuration.

Claim 3. *If X is k-bidismantlable, then* `Mark(`X`)` *marks all X-configurations.*

The general idea of our proof follows the proof of Theorem 12 of [27]. Let $\{v_1, \ldots, v_t\}$ be a k-bidismantling ordering of X. We prove by induction on $t - i$ that `Mark(`X_i`)` marks all X_i-configurations, where $X_i = \{v_i, \ldots, v_t\}$. The assertion trivially holds for X_{t-1}. Let $i < t - 1$. Assuming that all X_{i+1}-configurations are marked by `Mark(`X_{i+1}`)`, we prove that `Mark(`X_i`)` marks all X_i-configurations.

By definition of the k-bidismantling ordering, there exist two adjacent or coinciding vertices $x, y \in X_{i+1}$ such that $N_k(v_i, G \setminus \{x, y\}) \cap X_i \subseteq N_1(y)$. Roughly speaking, `Mark(`$X_i$`)` marks the X_i-configurations in the same order as `Mark(`X_{i+1}`)` marks the X_{i+1}-configurations, but once a configuration (c, y) with $c \in X_{i+1}$ is marked, `Mark(`X_i`)` also marks the configuration (c, v_i). Once `Mark(`X_i`)` has marked all X_i-configurations $(c, r) \in X_{i+1} \times X_i$, the remaining X_i-configurations (v_i, r) with $r \in X_{i+1}$ can also be marked by `Mark(`X_i`)`.

Let $\ell \geq 1$. By induction on ℓ, we prove that *any X_{i+1}-configurations (c, r) that is marked by* `Mark(`X_{i+1}`)` *with label at most ℓ will be also marked by* `Mark(`X_i`)`. Moreover, if $r = y$, we prove that *once* `Mark(`X_i`)` *has marked (c, r), then it can mark (c, v_i)*. Let us first prove this assertion for $\ell = 1$. For any $(c, r) \in X_i \times X_i$ with $r \in N_1(c)$, (c, r) is marked by `Mark(`X_i`)` with label 1 . If (c, y) is marked with label 1 (i.e., $y \in N_1(c) \cap X_i$), then (c, v_i) can be marked with 2. Indeed, for all $z \in N_k(v_i, G \setminus \{x, y\}) \cap X_i$, we have $z \in N_1(y)$ (by definition of the k-bidismantling order), and thus the X_i-configuration (y, z) is marked with label 1. Hence, by setting $(x_{(c,v_i)}, y_{(c,v_i)}) = (x, y)$, the procedure `Mark(`$X_i$`)` marks (c, v_i) with label 2.

Assume now that the induction hypothesis holds for some $\ell \geq 1$ and we will show that it still holds for $\ell + 1$. Let (c, r) be a X_{i+1}-configuration marked by `Mark(`X_{i+1}`)` with label $\ell + 1$. We first prove that (c, r) is eventually marked by `Mark(`X_i`)`. By definition of `Mark(`X_{i+1}`)`, there exist $y_{(c,r)} \in N_1(c) \cap X_{i+1}$ and $x_{(c,r)} \in (N_1(y_{(c,r)}) \setminus \{r\}) \cap X_{i+1}$ such that for all $z \in N_k(r, G \setminus \{x_{(c,r)}, y_{(c,r)}\}) \cap X_{i+1}$, the X_{i+1}-configuration $(y_{(c,r)}, z)$ is marked with label at most ℓ by `Mark(`X_{i+1}`)`. By the induction hypothesis, this implies that for all $z \in N_k(r, G \setminus$

$\{x_{(c,r)}, y_{(c,r)}\}) \cap X_{i+1}$, the X_{i+1}-configuration $(y_{(c,r)}, z)$ is marked by `Mark(`X_i`)`. If $v_i \notin N_k(r, G \setminus \{x_{(c,r)}, y_{(c,r)}\})$, then clearly (c, r) is marked by `Mark(`X_i`)`. Let us assume that $v_i \in N_k(r, G \setminus \{x_{(c,r)}, y_{(c,r)}\})$. We aim at proving that $(y_{(c,r)}, v_i)$ is eventually marked by `Mark(`X_i`)`. We distinguish three cases.

- If $y_{(c,r)} = y$, then $(y_{(c,r)}, v_i)$ is marked with label 1 since $y_{(c,r)} = y \in N_1(v_i)$.
- If $x_{(c,r)} = y$, then $(y_{(c,r)}, v_i)$ is marked with label 1 or 2 by setting $(x_{(y_{(c,r)},v_i)}, y_{(y_{(c,r)},v_i)}) = (x, y)$. Indeed, for all $z \in N_k(v_i, G \setminus \{x, y\}) \cap X_i$, we have $z \in N_1(y)$ (by definition of the k-bidismantling order), and thus the X_i-configuration (y, z) is marked with label 1.
- Otherwise, we assert that $(y_{(c,r)}, y)$ has already been marked by `Mark(`X_i`)`. By the induction hypothesis, this implies that $(y_{(c,r)}, v_i)$ was also marked.

  If $y \in N_k(r, G \setminus \{x_{(c,r)}, y_{(c,r)}\}) \cap X_{i+1}$ and since (c, r) is marked with label $\ell+1$ by the marking procedure in X_{i+1}, then $(y_{(c,r)}, y)$ must be marked by `Mark(`X_{i+1}`)` with label at most ℓ. By the induction hypothesis, this implies that $(y_{(c,r)}, y)$ has been marked by `Mark(`X_i`)`. Hence, it remains to show that $y \in N_k(r, G \setminus \{x_{(c,r)}, y_{(c,r)}\}) \cap X_{i+1}$.

  Let P be a path of length at most k between r and v_i in $G \setminus \{x_{(c,r)}, y_{(c,r)}\}$. If x or y belongs to P, then we trivially get that $y \in N_k(r, G \setminus \{x_{(c,r)}, y_{(c,r)}\}) \cap X_{i+1}$. Otherwise, this means that $r \in N_k(v_i, G \setminus \{x, y\}) \cap X_i$ and $r \in N_1(y)$ holds by definition of the bidismantling order. Hence, $y \in N_k(r, G \setminus \{x_{(c,r)}, y_{(c,r)}\}) \cap X_{i+1}$.

In all three cases, the pair $(y_{(c,r)}, v_i)$ is marked by `Mark(`X_i`)`. Thus, for all $z \in N_k(r, G \setminus \{x_{(c,r)}, y_{(c,r)}\}) \cap X_i$, the X_i-configuration $(y_{(c,r)}, z)$ has been marked. Therefore, this is also the case for the X_i-configuration (c, r). To conclude the proof, we need to show that, once a X_i-configuration (c, y) $(c \neq v_i)$ is marked by `Mark(`X_i`)`, then (c, v_i) can be marked as well. Since (c, y) has been marked, there exist $y_{(c,y)} \in N_1(c) \cap X_i$ and $x_{(c,y)} \in (N_1(y_{(c,y)}) \setminus \{y\}) \cap X_i$ such that for all $z \in N_k(y, G \setminus \{x_{(c,y)}, y_{(c,y)}\}) \cap X_i$, the X_i-configuration $(y_{(c,y)}, z)$ is marked. Let $z' \in N_k(v_i, G \setminus \{x_{(c,y)}, y_{(c,y)}\}) \cap X_i$. We prove that $z' \in N_k(y, G \setminus \{x_{(c,y)}, y_{(c,y)}\}) \cap X_i$, which shows that $(y_{(c,y)}, z')$ has been already marked. Let P be a shortest path between v_i and z' in $G \setminus \{x_{(c,y)}, y_{(c,y)}\}$. Note that $|P| \leq k$. If $y \in P$, clearly $z' \in N_k(y, G \setminus \{x_{(c,y)}, y_{(c,y)}\}) \cap X_i$. Else, if $x \in P$, then let P' be the subpath of P from z' to x. Then $P' \cup \{x, y\}$ is a path of length at most k between z' and y in the graph $G \setminus \{x_{(c,y)}, y_{(c,y)}\}$. Otherwise, $z' \in N_k(v_i, G \setminus \{x, y\}) \cap X_i$ and thus $z' \in N_1(y)$. Therefore, for any $z' \in N_k(v_i, G \setminus \{x_{(c,y)}, y_{(c,y)}\}) \cap X_i$, $(y_{(c,y)}, z')$ is marked and thus the pair (c, v_i) can be marked as well.

Summarizing, we conclude that for all $c, r \in X_{i+1}$, the configurations (c, r) and (c, v_i) are marked by the procedure `Mark(`X_i`)`. To conclude the proof, note that any configuration (v_i, r) can be marked as well: either with 1 if $r \in N_1(v_i)$ or by setting $(x_{(v_i,r)}, y_{(v_i,r)}) = (y, y)$ otherwise. $\square$

From Theorem 4 and by noticing that if a graph $G = (V, E)$ with n vertices is n-bidismantlable, then there are two vertices x, y such that y dominates a connected component of $G \setminus \{x, y\}$, we obtain the following observation:

Proposition 15. *$\mathcal{CWW}$ is the class of graphs which are k-bidismantlable for all $k \geq 1$.*

We continue with an example showing that k-bidismantlability is not a necessary condition for any $k \geq 3$.

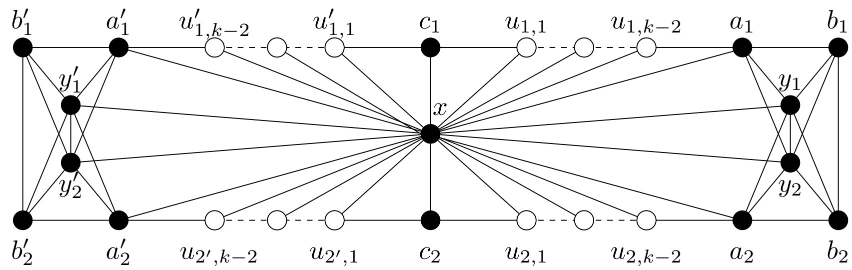


FIGURE 8. A graph $G \in \mathcal{CWW}(k)$ that is not k-bidismantlable

Proposition 16. *Let G be the graph from Fig. 8. Then $G \in \mathcal{CWW}(k)$, however G is not k-bidismantlable.*

Proof. To show that $G \in \mathcal{CWW}(k)$, we exhibit a strategy for $\mathcal{C}$. The cop starts at the vertex x. To avoid being captured immediately, the robber starts in b_1, b_2, b'_1, or b'_2, say in b_1 (the other cases are similar). Then the cop moves to a_1, goes to x and stays there during $k-2$ steps, and finally goes to y_1. Once $\mathcal{C}$ is in a_1, $\mathcal{R}$ can go to b_1, b_2, y_1 or y_2. Then, while $\mathcal{C}$ is in x, $\mathcal{R}$ can move to a vertex in $\{a_1, a_2, b_1, b_2, y_1, y_2\} \cup \{u_{1,i}, u_{2,i} : 2 \leq i \leq k-2\}$. Finally, when the cop moves to y_1, $\mathcal{R}$ can go to a vertex in $\{a_1, a_2, b_1, b_2, y_2, x\} \cup \{u_{1,i}, u_{2,i} : i \leq k-2\}$. Thus, if the robber is not catched the second time he shows up, he is in a vertex of $\{u_{1,i}, u_{2,i} : i \leq k-2\}$. If the robber shows up in a vertex of $\{u_{1,i} : i \leq k-2\}$ while the cop is in y_1 (the other case is similar), the cop oscillates between a_1 and x during k steps and finishes in x (if k is odd, $\mathcal{C}$ first moves to x and if k is even, $\mathcal{C}$ first moves to a_1). Thus, the next time $\mathcal{R}$ shows up, he is in a vertex of $\{c_1, a_1, a'_1\} \cup \{u_{1,i}, u'_{1,i} : i \leq k-2\}$: in any case, $\mathcal{C}$ (positioned at x) catches $\mathcal{R}$ when he shows up.

Now we show that G is not k-bidismantlable. We can eliminate a vertex v if there exist two neighbors $x(v), y(v)$ such that $N_k(v, G \setminus \{x, y\}) \cap X \subseteq N_1(y)$, where X is the set of vertices that have not been yet eliminated. First note that for any $i \leq k-2$, we can eliminate $u_{1,i}$ with $x(u_{1,i}) = a_{1,i}$ and $y(u_{1,i}) = x$. By symmetry, we can also eliminate $u_{2,i}$, $u'_{1,i}$ and $u'_{2,i}$ for any $i \leq k-2$ (these vertices are colored white in Fig. 8). We show that no other vertex can be eliminated after a set $Y \subseteq \{u_{1,i}, u'_{1,i}, u_{2,i}, u'_{2,i} : i \leq k-2\}$ of vertices has been eliminated (Y can be empty or contain all these vertices).

Let $X = V(G) \setminus \{u_{1,i}, u_{2,i}, u'_{1,i}, u'_{2,i} : i \leq k-2\}$. By symmetry, it is sufficient to show that for any $v \in \{a_1, b_1, c_1, x, y_1\}$, and any adjacent vertices $x(v), y(v) \in V(G)$, there exists $z(v) \in (N_k(v, G \setminus \{x(v), y(v)\}) \cap X) \setminus N_1(y(v))$. For any v and $y(v)$, this condition is true as soon as $N_1(v) \cap X \not\subseteq N_1(y(v))$.

If $v = x$, then there does not exists any $y(v)$ such that $N_1(v) \cap X \subseteq N_1(y(v))$. If $v = a_1$ and $N_1(v) \cap X \subseteq N_1(y(v))$, then $y(v) \in \{y_1, y_2\}$; in both cases, $x(v) \notin \{u_{1,i} : i \leq k-2\}$

and thus $z(v) = c_1$ satisfies the condition. If $v = b_1$ and $N_1(v) \cap X \subseteq N_1(y(v))$ then $y(v) \in \{y_1, y_2\}$; by symmetry, we assume $y(b_1) = y_1$. Note that there exist two vertex-disjoint paths $(b_1, a_1, u_{1,k-2}, \ldots, u_{1,1}, c_1)$ and (b_1, y_2, x, c_1) of length at most k from b_1 to c_1 avoiding y_1. Consequently, for any choice of $x(b_1)$, the vertex $z(v) = c_1$ satisfies the condition. If $v = y_1$ and $N_1(v) \cap X \subseteq N_1(y(v))$, then $y(v) = y_2$. Again, there exist two vertex-disjoint paths $(y_1, a_1, u_{1,k-2}, \ldots, u_{1,1}, c_1)$ and (y_1, x, c_1) of length at most k from y_1 to c_1 avoiding y_2. Hence, for any choice of $x(y_1)$, the vertex $z(v) = c_1$ satisfies the condition. Finally, suppose that $v = c_1$ and $N_1(v) \cap X \subseteq N_1(y(v))$. Then $y(v) \in \{u_{1,1}, u'_{1,1}, x\}$. If $y(v) = u_{1,1}$ (the case $y(v) = u'_{1,1}$ is similar), then $x(v) \notin \{a'_1\} \cup \{u'_{1,i} : i \leq k-2\}$ and thus $z(v) = b_1$ satisfies the condition. If $y(v) = x$, then either $x(v) \notin \{a'_1\} \cup \{u'_{1,i} : i \leq k-2\}$, or $x(v) \notin \{a_1\} \cup \{u_{1,i} : i \leq k-2\}$. By symmetry, we assume $x(v) \notin \{a'_1\} \cup \{u'_{1,i} : i \leq k-2\}$; in this case, $z(v) = b_1$ satisfies the condition. $\square$

Open question 3: Characterize the k-winnable graphs for $k = 2, 3$ and, more generally, for all k.

5.3. **Cop-win graphs for game with fast robber and witness.** We now consider a variant of the game where the robber is visible every k moves and has speed s while the cop has speed 1. It means that at each step, the robber can move to a vertex at distance at most s from his current position, and that the cop can see the robber only every k steps. We denote by $\mathcal{CWFRW}(s, k)$ the class of graphs where a single cop with speed 1 can catch a robber with speed s that is visible every k moves. By definition, we have $\mathcal{CWFRW}(1, k) = \mathcal{CWW}(k)$ and $\mathcal{CWFRW}(s, 1) = \mathcal{CWFR}(s)$.

Theorem 6. *If $s \geq 3, k \geq 1$ or $s \geq 2, k \geq 2$, then $\mathcal{CWFRW}(s, k)$ is the class of big brother graphs.*

Proof. We know from Theorem 3 that if $s \geq 3$ and $k \geq 1$, $\mathcal{CWFR}(sk) = \mathcal{CWFR}(s)$ is the class of big brother graphs. Consequently, since $\mathcal{CWFR}(sk) \subseteq \mathcal{CWFRW}(s, k) \subseteq \mathcal{CWFR}(s)$, it follows that $\mathcal{CWFRW}(s, k)$ is the class of big brother graphs for all $s \geq 3$ and $k \geq 1$.

In the remaining of this proof, we show that when $s = 2$ and $k \geq 2$, $\mathcal{CWFRW}(2, k)$ also coincides with the class of big brother graphs. This proof follows closely the proof of Theorem 4. In particular, the following proposition is the counterpart of Proposition 9.

Proposition 17. *Let $G \in \mathcal{CWFRW}(2, k)$ for $k \geq 2$. If the minimum degree of G is at least 2, then G contains two vertices v and y such that $N_{2k}(v, G \setminus \{y\}) \subseteq N_1(y)$.*

Proof. If G contains a dominating vertex y, then the result holds for any $v \neq y$. Assume thus that G has no dominating vertices. Consider a parsimonious winning strategy of the cop and suppose that the robber uses a strategy to avoid being captured as long as possible. Since G does not contain leaves, the robber is caught immediately after having been visible. Since G does not have any dominating vertex, the robber is visible at least twice. Let y be the vertex occupied by the cop when the robber becomes visible for the last time before his capture. Let v be the next-to-last visible vertex occupied by the robber. Finally, let

$S_c = (c_0, c_1, \ldots, c_k = y)$ be the trajectory of the cop during the last k steps (repetitions are allowed). Note that $v \notin N_1(c_0)$, otherwise the robber would have been caught immediately. We distinguish two cases depending of whether or not $c_i = y$ for all $1 \leq i \leq k$.

If for any $1 \leq i \leq k$, we have $c_i = y$, then $\mathcal{R}$ could have move to any vertex $w \in N_{2k}(v, G \setminus \{y\})$. Since the trajectory of the robber is maximal, the robber is caught in any such w and thus $N_{2k}(v, G \setminus \{y\}) \subseteq N_1(y)$. Suppose now that there exists i such that $c_i \neq y$. Let i be the largest index such that $c_i \neq y$.

Claim 4. *If G contains a cycle C and a vertex $w \in C$ such that $d(v, w) < d(c_1, w)$, then $G \setminus \{y\}$ has a connected component that is dominated by y.*

Proof. Let w be the closest to v vertex satisfying the condition of the claim. If the assertion of the claim is not satisfied, we will exhibit a strategy allowing the robber to escape the cop during more steps, contradicting the choice of the trajectory $\mathcal{R}$. Suppose that at the beginning of the last phase, the robber moves from v to w along a shortest (v, w)-path. Since $d(v, w) < d(c_1, w)$, the robber cannot be intercepted by the cop during these moves. Suppose that the robber reaches the vertex w before the ith step when the cop arrives at c_i. Then by Lemma 1 (adapted to this game) the robber can safely move on C until the cop reaches the vertex c_i.

In both cases, let z be the current position of the robber when the cop reaches c_i. Then $z \in N_1(y)$, otherwise the robber can remain at z without being caught because starting with this step the cop remains in y. If $z \neq y$, then let $u = z$, and if $z = y$, let u be a neighbor of y different from c_i (it exists because the minimum degree of a vertex of G is at least 2). In both cases, let H be the connected component of $G \setminus \{y\}$ that contains u. We assert that y dominates all the vertices of H. Suppose this is not the case and consider a vertex t in $V(H) \setminus N(y)$ that is at a minimum distance from u in H. From our choice of t, we can find a common neighbor $r \in V(H)$ of y and t. If $r \neq c_i$, then while the cop is in c_i, the robber can go from z (z is either u or y) to r through y and then, when $\mathcal{C}$ goes to y, $\mathcal{R}$ goes to t and stays there until he becomes visible. If $r = c_i$, then let s be a neighbor of r on a shortest (u, r)-path in $G \setminus \{y\}$. By our choice of t, necessarily $s \in N(y)$. Thus, when the cop is in c_i, the robber can go from z to s through y. And then, when the cop goes to y, $\mathcal{R}$ goes to t through r and stays there until he becomes visible. In both cases, by following such a strategy, $\mathcal{R}$ could avoid being caught. This contradicts the maximality of the trajectory of the robber. This concludes the proof of the claim. □

We now complete the proof of Proposition 17. If the vertex v belongs to a cycle C, then setting $w := v$ and applying Claim 4, we conclude that y dominates a non-empty connected component H of $G \setminus \{y\}$ establishing thus the assertion. So, suppose that v is an articulation point of G not contained in a cycle. Since the minimum degree of G is at least 2, $G \setminus \{v\}$ has a connected component H that does not contain c_0 (nor c_1). Necessarily H contains a cycle C, otherwise we will find in H a vertex of degree 1 in G. Since any path from c_1 to a vertex w of C goes through v, we obtain $d(v, w) < d(c_1, w)$. Then, the result again follows from Claim 4. This ends the proof of Proposition 17. □

Finally, we prove Theorem 6 when $s = 2$ and $k \geq 2$. Consider a graph $G \in \mathcal{CWFRW}(2, k)$. To establish that G is a big brother graph, in view of Theorem 3, it suffices to show that G is $(2k, 1)$-dismantlable. For this, by Proposition 6, we just have to show that there exists an ordering $v_1, \ldots, v_n$ of the vertices of G such that for each $1 \leq i < n$ there exists $y \in \{v_{i+1}, \ldots, v_n\}$ such that $N_{2k}(v_i, G_i \setminus \{y\}) \subseteq N_1(y, G_i)$. We proceed by induction on the size of G. Suppose that G has at least two vertices, otherwise the result is trivial. If G has a vertex v of degree 1, then let y be the unique neighbor of v in G. In this case, then obviously $N_{2k}(v, G \setminus \{y\}) \subseteq N_1(y, G)$. Otherwise, by Proposition 17, we can find vertices v and y such that $N_{2k}(v, G \setminus \{y\}) \subseteq N_1(y, G)$.

We now show that $G' = G \setminus \{v_1\}$ also belongs to $\mathcal{CWFRW}(2, k)$. Consider a winning positional strategy σ for the cop in G. As in the proof of Theorem 1, we define a strategy σ' for the cop in G' using one bit of memory. Starting from σ, we define $\sigma'(c, r, m)$ for any positions $c, r \in V(G')$ of the cop and the robber and for any value of the memory $m \in \{0, 1\}$. The idea is that the cop plays using σ except when he is in y and his memory contains 1; in this case he uses σ as if he was in v (going via y instead of v if v appears in his sequence of moves).

If $m = 0$ or $c \neq y$, let $\sigma(c, r) = (c_1, \ldots, c_k)$ and for each i, let $c'_i = c_i$ if $c_i \neq v$ and $c'_i = y$ otherwise (this is possible since $N_1(v) \subseteq N_1(y)$). If $c_k = v$, then $\sigma'(c, r, m) = ((c'_1, \ldots, c'_{k-1}, y), 1)$, otherwise let $\sigma'(c, r, m) = ((c'_1, \ldots, c'_{k-1}, c_k), 0)$.

If $m = 1$ and $c = y$, let $\sigma(v, r) = (c_1, \ldots, c_k)$ and for each i, let $c'_i = c_i$ if $c_i \neq v$ and $c'_i = y$ otherwise. If $c_k = v$, then $\sigma'(y, r, 1) = ((c'_1, \ldots, c'_{k-1}, y), 1)$, otherwise let $\sigma'(y, r, 1) = ((c'_1, \ldots, c'_{k-1}, c_k), 0)$.

Let $S_r = (r_1, r_2, \ldots, r_p, \ldots)$ be a valid sequence of moves in G'. Since $V(G') \subseteq V(G)$, S_r is also a valid sequence of moves in G. Let $S_c = (c_1, \ldots, c_p, \ldots)$ be the corresponding valid sequence of moves of the cop playing σ against S_r in G and let $S'_c = (c'_1, \ldots, c'_p, \ldots)$ be the valid sequence of moves of the cop playing σ' against S_r in G'. Note that the sequences of moves S_c and S'_c differ only if $c_k = v$ and $c'_k = y$. Finally, since the cop follows a winning strategy for G, there exists a step j such that $c_j = r_j \in V(G')$ (note that $r_j \neq v$ because we supposed that S_r is a valid sequence of moves in G'). Since $c_j \neq v$, we also have $c'_j = r_j$, thus $\mathcal{C}$ captures $\mathcal{R}$ in the game restricted to G'. In conclusion, starting from a positional strategy for the game in G, we have constructed a winning strategy using memory for the game in G'. As mentioned in the introduction, it implies that there exists a positional winning strategy for the game in G'. Consequently, $G' \in \mathcal{CWFRW}(2, k)$ and by induction hypothesis, G' has $(2k, 1)$-dismantling order $(v_2, \ldots, v_n)$, whence $(v, v_2, \ldots, v_n)$ is a $(2k, 1)$-dismantling order of G. This concludes the proof of Theorem 6. $\square$

6. Bipartite cop-win graphs for game with "radius of capture"

In this section we characterize bipartite graphs of the class $\mathcal{CWRC}(1)$, i.e., the bipartite cop-win graphs in the cop and robber game with radius of capture 1. Recall that in this game introduced in [11], $\mathcal{C}$ and $\mathcal{R}$ move at unit speed and the cop wins if after his move he is within distance at most 1 from the robber. Notice that any graph of diameter 2 belongs

to $\mathcal{CWRC}(1)$: given the positions u and v of the cop and the robber, to capture the robber the cop simply moves from u to a common neighbor of u and v.

Following [7], a bipartite graph G is called *dismantlable* if its vertices can be ordered $v_1, \ldots, v_n$ so that $v_{n-1}v_n$ is an edge of G and for each $v_i, i < n-1$, there exists a vertex $y := v_j$ with $j > i$ (necessarily not adjacent to v_i) such that $N(v_i, G_i) := N_1(v_i, G_i) \setminus \{v_i\} \subseteq N_1(y)$. Note that for any i, G_i is a retract of G and therefore an isometric subgraph of G.

Theorem 7. *A bipartite graph G belongs to $\mathcal{CWRC}(1)$ if and only if G is dismantlable.*

Proof. First suppose that $G \in \mathcal{CWRC}(1)$. If G has diameter 2, then necessarily G is a complete bipartite graph, which is obviously dismantlable. Suppose now that G has diameter at least 3. As in previous proofs of similar results, we assume that $\mathcal{C}$ uses a parsimonious strategy. Consider a maximal sequence of moves of the robber before he get caught. Let v be the next-to-last position of the robber and let y be the position of the cop at this step (y is not adjacent to v, otherwise $\mathcal{C}$ captures $\mathcal{R}$ in v). This means that for any $w \in N_1(v)$, the cop can move in some vertex $u \in N_1(y)$ such that $w \in N_1(u)$. This shows that $N_1(v) \subseteq N_2(y)$. Since G is bipartite, this means that $d(v, y) = 2$ and all neighbors of v are adjacent to y, i.e., $N(v) \subseteq N(y)$.

We now show that $G' = G \setminus \{v\}$ also belongs to $\mathcal{CWRC}(1)$. Consider a winning positional strategy σ for the cop in G. As in the proof of Theorem 1, we define a strategy σ' for the cop in G' using one bit of memory. Starting from σ, we define $\sigma'(c, r, m)$ for any positions $c, r \in V(G')$ of the cop and the robber and for any value of the memory $m \in \{0, 1\}$. The idea is that the cop plays using σ except when he is in y and his memory contains 1; in this case he uses σ as if he was in v (going to y instead of v). If $m = 0$ or $c \neq y$, if $\sigma(c, r) = v$ then $\sigma'(c, r, m) := (y, 1)$ (this is a valid move since $N(v) \subseteq N(y)$ and $c \neq v$), otherwise let $\sigma'(c, r, m) := (\sigma(c, r), 0)$. If $m = 1$ and $c = y$, if $\sigma(v, r) = v$, then $\sigma'(y, r, 1) := (y, 1)$, otherwise let $\sigma'(y, r, 1) := (\sigma(v, r), 0)$.

Let $S_r = (r_1, r_2, \ldots, r_p, \ldots)$ be a valid sequence of moves in G'. Since $V(G') \subseteq V(G)$, S_r is also a valid sequence of moves in G. Let $S_c = (c_1, \ldots, c_p, \ldots)$ be the corresponding valid sequence of moves of the cop playing σ against S_r in G and let $S'_c = (c'_1, \ldots, c'_p, \ldots)$ be the valid sequence of moves of the cop playing σ' against S_r in G'. Note that the sequences of moves S_c and S'_c differ only if $c_k = v$ and $c'_k = y$. Finally, since the cop follows a winning strategy for G, there exists a step j such that $c_{j+1} \in N_1(r_j) \subseteq V(G')$ (note that $r_j \neq v$ because we supposed that S_r is a valid sequence of moves for the game in G'). Since $N(v) \subseteq N(y)$, we also have $c'_{j+1} \in N_1(r_j)$, thus $\mathcal{C}$ captures $\mathcal{R}$ in the game restricted to G'. In conclusion, starting from a positional strategy for the game in G, we have constructed a winning strategy using memory for the game in G'. As mentioned in the introduction, it implies that there exists a positional winning strategy for the game in G'. Consequently, $G' \in \mathcal{CWRC}(1)$ and by induction hypothesis, G' has a dismantling order $(v_2, \ldots, v_n)$, whence $(v, v_2, \ldots, v_n)$ is a dismantling order of G.

Conversely, suppose that a bipartite graph G is dismantlable and let $v = v_1, v_2, \ldots, v_n$ be a dismantling order of G. If G has diameter 2, then G is a complete bipartite graph and thus,

$G \in \mathcal{CWRC}(1)$. Suppose now that G has a diameter at least 3. By induction hypothesis, $G' = G(\{v_2, \ldots v_n\})$ belongs to $\mathcal{CWRC}(1)$. Suppose that v is dominated by a vertex y at distance 2 from v.

Consider a parsimonious positional winning strategy σ' for the cop in G'. Using σ', we build a parsimonious positional winning strategy σ for the cop in G. As in the previous proofs, the idea is that if $\mathcal{C}$ sees $\mathcal{R}$ in v, he plays as in the game on G' when the cop is in y. For any positions $c, r \in V(G)$, if $r \in N_2(c)$, then $\sigma(c,r) := u \in N_1(c) \cap N_1(r)$. Otherwise, if $c \in N(y) \setminus N(v)$ and $r = v$, then $\sigma(c,v) := y$. Otherwise, if $c, r \neq v$, then $\sigma(c,r) := \sigma'(c,r)$; if $r = v$ and $c \notin N_2(v)$, then $\sigma(c,v) := \sigma'(c,y)$; finally, if $c = v$ and $r \notin N_2(v)$, then $\sigma(v,r) := u \in N(v)$ (in fact, if the cop plays according to σ, he will never move to v). By construction, σ is parsimonious and positional; in particular, $\sigma(y,v) \in N(v)$.

We now prove that σ is a winning strategy. Consider any valid sequence $S_r = (r_1, \ldots, r_k, \ldots)$ of moves of the robber in G, and let $S'_r = (r'_1, \ldots, r'_k, \ldots)$ be the sequence obtained by setting $r'_k = r_k$ if $r_k \neq v$ and $r'_k = y$ if $r_k = v$. Since $N(v) \subseteq N(y)$, S'_r is a valid sequence of moves for the robber in G'. By induction hypothesis, for any initial position of $\mathcal{C}$ in G', the strategy σ' enables $\mathcal{C}$, following a trajectory $S'_c = (c'_1, \ldots, c'_k, \ldots)$, to catch $\mathcal{R}$ which moves according to S'_r, i.e., there exists an index m such that $r'_m \in N_1(c'_{m+1})$. Suppose that $\mathcal{C}$ chooses his starting position c_1 in G' and that the cop, following σ, plays in G the sequence $S_c = (c_1, \ldots, c_k, \ldots)$ against the sequence S_r of the robber. Note that $c_k = c'_k$ for any $k < m$, i.e., except when the robber is caught in G'. If $r_m = r'_m$, then $r_m \in N_1(c_{m+1})$ and $\mathcal{C}$ captures $\mathcal{R}$. Now suppose that $r_m \neq r'_m$. From the definition of S'_r we conclude that $r'_m = y$ and $r_m = v$. If $v \in N(c_{m+1})$, the robber is caught at step $m+1$. Otherwise, since $c_{m+1} \in N(y) \setminus N(v)$, then $c_{m+2} = y$. If at step r_{m+1} the robber moves to a neighbor of v, since $N(v) \subseteq N(y)$, we conclude that $r_{m+1} \in N(c_{m+2})$ and the robber is caught. Finally, if the robber remains at v (i.e., $r_{m+1} = v$), then to avoid being caught while moving, at step $m+2$, $\mathcal{R}$ must also stay in v and then at step $m+3$ the cop moves from y to any common neighbor of y and v and catches the robber. □

In the classical game of cop and robber where both players have speed 1 and there is no radius of capture, a bipartite graph G is cop-win if and only if G is a tree. The previous result shows that in the variant of the game with a radius of capture, a single cop can win in a considerably larger class of graphs.

Bonato and Chiniforooshan [11] asked for a characterization of graphs of $\mathcal{CWRC}(1)$. Theorem 7 answers the question in the case of bipartite graphs. On the other hand, characterizing the graphs of $\mathcal{CWRC}(1)$ using a specific dismantling scheme seems to be quite challenging. Two natural candidates for us were the total orders $v_1, \ldots, v_n$ satisfying the following conditions:

(i) for each vertex v_i, there exists a vertex $v_j, j > i$, such that $N_1(v_i, G_i) \subseteq N_2(v_j, G_{i+1})$;
(ii) for each vertex v_i, there exists a vertex $v_j, j > i$, such that $N_2(v_i, G_i) \subseteq N_2(v_j, G_i)$.

It seems that the first condition is necessary, while the second condition is sufficient. However, we were not able to prove this. In fact, one can easily show that any graph $G \in \mathcal{CWRC}(1)$

contains two vertices v, y such that $N_1(v, G_i) \subseteq N_2(y, G_{i+1})$, but we cannot show that $G \setminus \{v\}$ also belongs to $\mathcal{CWRC}(1)$. A similar difficulty occurs while establishing the sufficiency of the second dismantling order.

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